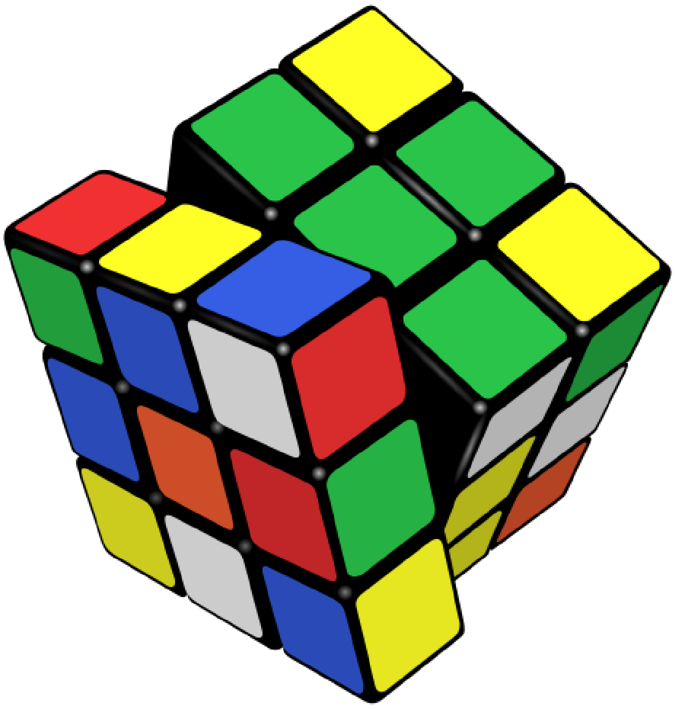
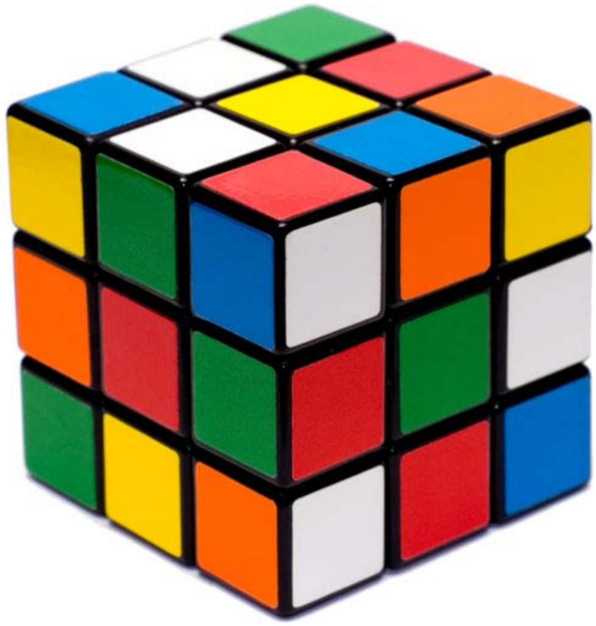
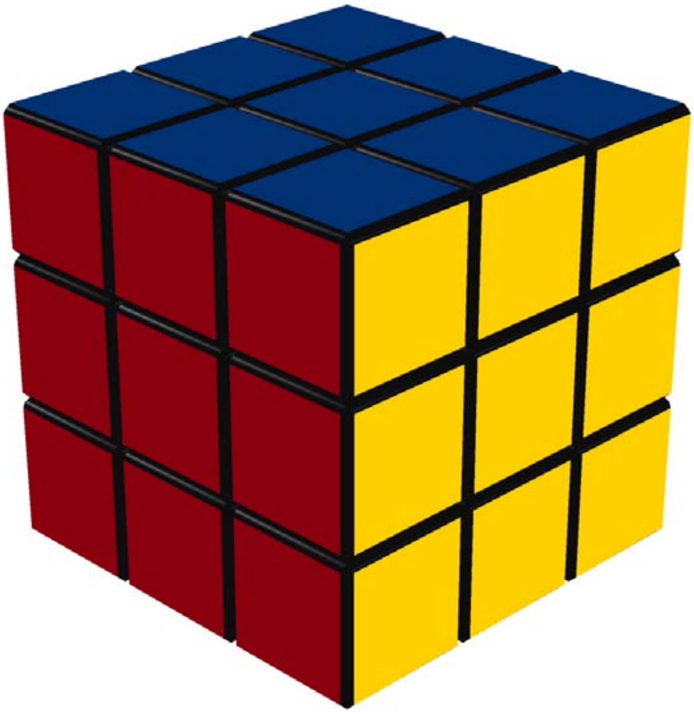
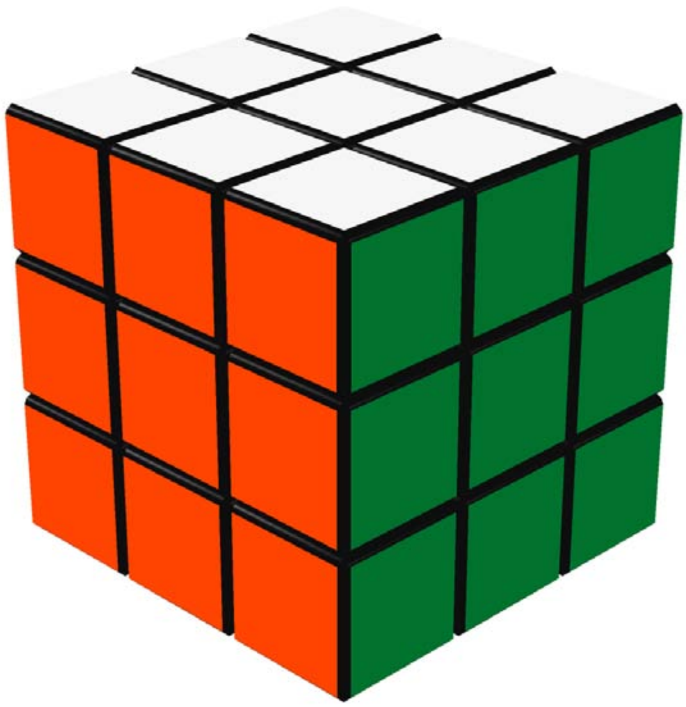
# Rubik’s Cube Solutions

| Rubik’s Cube Solution – Useful Links | |
|---|---|
| http://www.geocities.com/jaapsch/puzzles/theory.htm | http://www.ryanheise.com/cube/ |
| http://peter.stillhq.com/jasmine/rubikscubesolution.html | |
| http://en.wikibooks.org/wiki/How_to_solve_the_Rubik's_Cube | |
| http://www.rubiks.com/World/~/media/Files/Solution_book_LOW_RES.ashx | |
| http://helm.lu/cube/MarshallPhilipp/index.htm | |
| **Rubik’s Cube in a Scrambled State** | |
|  |  |
| **Rubik’s Cube in a Solved State – CubeTwister** | |
|  |  |
| Front: Red, Right: Yellow, Up: Blue | Back: Orange, Down: Green, Left: White |
| Cube Colors: Red opposed to Orange, Yellow opposed to White, Blue opposed to Green | |

# Commutators and Conjugates

## Introduction

A Commutator is an algorithm of the form X Y X' Y', and a conjugate is an algorithm of the form X Y X', where X and Y denote arbitrary algorithms on a puzzle, and X', Y' denote their respective inverses. They are formal versions of the simple, intuitive idea of "do something to set up another task which does something useful, and undo the setup." Commutators can be used to generate algorithms that only modify specific portions of a cube, and are intuitively derivable. Many puzzle solutions are heavily or fully based on commutators.

## Commutator and Conjugate Notation

[X, Y] is a commonly used notation to represent the sequence X Y X' Y'. [X: Y] is a well-accepted representation of the conjugate X Y X'.
Since commutators and conjugates are often nested together, Lucas Garron has proposed the following system for compact notation: Brackets denote an entire algorithm, and within these, the comma delimits a commutator, and a colon or a semicolon a conjugate. The symbols are given order of precedence: colon, comma, semicolon. For example, [X; Y, Z: W] represents X ( Y (Z W Z') Y' (Z W' Y')) X'.

## Corner Twists

On the Rubik's Cube, commutators are applied for swapping and twisting corners and edges. Commutation works best when X and Y are nearly disjoint. Therefore, let us choose Y to be a turn of the Down Face (D) and X so that it affects only a *single* piece in the Down Face (D).

An extremely useful choice of X is the **monotwist** X = L' U L F U F'. This twists only *one* corner (DLF) and does not affect anything else in the D Layer. The upper-half of the cube is messed up by the **monotwist** but this does not matter because of the commutation. We now have three very useful sequences for twisting *two* corners without affecting anything else on the cube:

X Y X' Y' = (L' U L F U F') D (L' U L F U F')' D' = (DLF)+ (DBL)- (to twist *adjacent* corners in the D Layer)

X Y2 X' Y2 = (L' U L F U F') D2 (L' U L F U F')' D2 = (DLF)+ (DRB)- (to twist *opposed* corners in the D Layer)

X Y' X' Y = (L' U L F U F') D' (L' U L F U F')' D = (DLF)+ (DFR)- (to twist *adjacent* corners in the D Layer)

## Solution 1: Layer-by-Layer (See [Jaap's Puzzle Page])

### Notation

Let the faces be denoted by letters L, R, F, B, U and D (Left, Right, Front, Back, Up and Down). Clockwise quarter turns of a face are denoted by the appropriate letter, anti-clockwise quarter turns by the letter with an apostrophe (i.e. L', R', F', B', U' or D'). Half turns are denoted by a letter followed by number 2 (i.e. L2, R2, F2, B2, U2 or D2).

### Phase 1: Solve Top Layer Edges (First Cross)

Find an edge piece that belongs to the Top Layer but which is not already located there. If there is an edge located on the Top Layer but not positioned correctly, then rotate the side (half-turn) containing this edge to place it in the Bottom Layer.

Rotate the Bottom Layer to place the piece *just below* its destination location, and then hold the cube so that *both* the piece and its destination location are on the Front Face.

1- To move DF to UF, do F2
2- To move FD to UF *without* disturbing other First Layer edges, do U' F' R F U

If the edge is in the Middle Layer, then hold the cube so that the piece is at the Front Right side, and do one of the following:

1- To move RF to UF, do F'
2- To move RF to UR, do U F' U'
3- To move RF to UB, do U2 F' U2
4- To move RF to UL, do U' F' U
5- To move FR to UF, do U' R U
6- To move FR to UR, do R
7- To move FR to UB, do U R U'
8- To move FR to UL, do U2 R U2

With experience you can save many turns of the U Face and choose the *order* in which you solve the edges so that this phase should usually takes no more than 7 or 8 moves in total.

### Phase 2: Solve Top Layer Corners

Find a corner piece in the Bottom Layer that belongs to the Top Layer. If there is a corner already located on the Top Layer but not positioned correctly, then hold the cube so that this corner is located at UFR.

If the corner piece front color is the same as the Front Face color, then do R' D R otherwise do R' D' R.

Rotate the Bottom Layer to place the corner piece below its destination, and hold the cube so that the piece and its destination location are on the Front Right side. Then do one of the following:

1- To move FRD to URF, do F D F'
2- To move RDF to URF, do R' D' R
3- To move DFR to URF, do F D' F' R' D2 R

By *first* solving corners which do *not* display the U Face color on the D Face, the longer sequence of case 3 can often be avoided.

### Phase 3: Solve Middle Layer Edges

Find an edge piece in the Bottom Layer that belongs to the Middle Layer. If there is an edge already located in the Middle Layer and if it is not positioned correctly, then choose any other *valid* edge from the bottom edges to displace this edge from the Middle Layer down to the Bottom Layer. Hold the cube so that the edge destination place is located at the Front Right side, then rotate the Bottom Layer to move the edge piece on the Front Face.

Do one of the following to place the edge correctly:

1- To move FD to FR, do D' R' D R D F D' F'
2- To move DF to FR, do D2 F D' F' D' R' D R

## Phase 4: Position Bottom Corners

Rotate the Bottom Layer until *at least* two corners are positioned correctly, ignoring their orientations.

If you need to swap two corners, then do one of the following:

1- To swap DLF and DFR, do R' D' R F D F' R' D R D2
2- To swap DLF and DRB, do R' D' R F D2 F' R' D R D

## Phase 5: Orient Bottom Corners

1- If *four* corners are twisted, then hold the cube so that there is one *clockwise* twisted corner on the Front Left side (the D Face color is displayed on the *left* side of this corner).

   If there are *three* twisted corners, then hold the cube so that the corner which is *not* twisted is located on the Front Left side (the D Face color is displayed on the *bottom* side of this corner).

   If there are *two* twisted corners, then hold the cube so that the corner which is *anticlockwise* twisted is located on the Front Left side (the D Face color is displayed on the *front* side of this corner)

2- Perform R' D' R D' R' D2 R D2

3- Repeat steps 1-2 until all four corners are correctly oriented.

If there are only *two* twisted corners after Phase 4, there is a *shorter* alternative using a monotwist commutator as follows.

If there are *two adjacent* twisted corners then hold the cube so that the corner which is twisted *anti-clockwise* is located on the Front Left side (the D Face color is displayed on the *front* side of this corner).

Perform (L' U L F U F') D (L' U L F U F')' D'

If there are *two opposed* twisted corners then hold the cube so that the corner which is twisted *anti-clockwise* is located on the Front Left side (the D Face color is displayed on the *front* side of this corner).

Perform (L' U L F U F') D2 (L' U L F U F')' D2

## Phase 6: Position Bottom Edges

Do one of the following:

1- To swap DL-DR and DF-DB, do L2 R2 U2 L2 R2 D L2 R2 U2 L2 R2 D'
2- To swap DF-DR and DL-DB, do R2 L2 U F2 R2 L2 B2 R2 L2 U' R2 L2
3- To cycle DR->DB->DL->DR, do L' R F L R' D2 L' R F L R'
4- To cycle DL->DB->DR->DL, do L' R F' L R' D2 L' R F' L R'

You may not need to use sequences 1 and 2 because edges can be solved by applying sequences 3 or 4 *twice*.

## Phase 7: Orient Bottom Edges

Do one of the following:

1- To flip DF, DR, do F U' D R2 U2 D2 L D' L' D2 U2 R2 D' U F' D
2- To flip DF, DB, do F U' D R2 U2 D2 L D2 L' D2 U2 R2 D' U F' D2
3- To flip *all* four edges, apply either of the above sequences *twice*

## Phase 8: Orient Face Centers

This phase is only necessary for picture cubes or for Rubik's World cube, where face centers have a visible orientation.

If two face centers need twisting, then hold the cube such that one of the centers is on located the Top Layer and the other one is located either on the Bottom layer or on the Right face. Then do one of the following:

1- To turn centers U and R', do R L' F B' U D' R' U' D F' B R' L U
2- To turn centers U2 and R2, do R L' F B' U D' R2 U' D F' B R' L U2
3- To turn centers U' and R, do R L' F B' U D' R U' D F' B R' L U'
4- To turn centers U and D', do R L' F2 B2 R L' U R L' F2 B2 R L' D'
5- To turn centers U2 and D2, do R L' F2 B2 R L' U2 R L' F2 B2 R L' D2
6- To turn centers U' and D, do R L' F2 B2 R L' U' R L' F2 B2 R L' D

Note that it is possible to take care of the top and side centers during the first 7 phases, so that at most only the Bottom Face center needs twisting. This step is then unnecessary.

If any face center needs a half turn, then hold the cube with that face on the top, and do the following:

To turn center U2, do R L U2 R' L' U R L U2 R' L' U.

# Other Solutions

There are many similar solutions that solve the cube in layers. Many are a lot faster than this one. Speed can be improved in many ways:

- Use more sequences for orienting the final corners, and in general use shorter sequences for phases 4-7.

- Phase 2 and 3 can be combined, so that top corners and middle edges are first placed adjacent in the bottom layer and then slotted into position as a single unit.

- Phase 2 and 3 can be combined in a different way: First solve 3 top corners. Then place the three adjacent middle edges correctly, which uses short move sequences because of the unsolved corner. Each edge is done by turning the U face to place the unsolved corner above the destination edge, and then using three (or four) moves to place the edge. Once these 3 edges are done, the final corner/edge pair is put together and solved as a unit.

- It is also possible to combine phase 1 with 2 and 3, by first building a 2×2×2 block, then a 2×2×3 block, and only then the last edge of the Top Layer. See Lars Petrus' website.

- Phase 4-7 can be combined in several different ways, nearly all of them involve memorizing a large number of sequences.

- You could position all the bottom pieces correctly first (23 sequences), and then orient them (57 sequences). See the book 'Winning Ways Vol.2' by Berlekamp, Conway and Guy.

- You could do this in the opposite order too, which is a bit better: orient them first (57 sequences) and then position them (23 sequences). This is better because it is easier to recognize the situation in this order. See for example Jessica Fridrich's website.

- You could solve corners first (42 sequences), then edges (29 sequences).

- You could just do this in the opposite order too: edges first (21 sequences), then corners (86 sequences), but there are rather many sequences needed to do corners in one step. You could however position (4) and orient (7) corners in two steps (in either order) while leaving edges untouched.

- Another approach is that used by Lars Petrus on his web pages. He solves the two top layers except one corner/edge pair. He then uses the unsolved column for orienting edges. After that, he solves the column, and the final layer like in phases 4-6, except that all the sequences he uses do not change edge orientations. His method seems to be the only fast solution that does not involve a large amount of memorization.

## Solution 2: Corners First (See [Jaap's Puzzle Page]())

Below is a corners first solution, i.e. all corners are solved first, then edges. Conceptually this is a good idea, since slice moves can be used (a slice move is a middle layer move), which only involves edges. This kind of method is relatively easy to understand and remember. It can be quite fast, but generally uses more moves than a layer method because slice turns are often counted as two moves.

### Further Notation

The only slice moves used in the following solution are moves of the middle layer M, i.e. the layer between U and D. Let M (or MU) denotes a clockwise quarter turn of the middle layer when looking from above, i.e. in the same direction as the U move.

### Phase 1: Solve Corners

Use any method you like. Use parts of the Layer-by-Layer Solution, or use the method of the mini-cube. Afterwards the top and bottom centers should match their corners, but the centers in the Middle Layer need not match.

### Phase 2: Solve Edges in U/D Layers

a. Find the edge piece that belongs to the top of the Front Face. If this edge is not found in the Middle Layer, then rotate the whole cube around the U/D axis to bring it to the Front Face and do F M2 F'.
b. Hold the cube with the destination location of the piece at the top of the Front Face and turn M to bring the piece to the Back Right. Then do one of the following:
   1. To move BR to UF, do F M F'
   2. To move RB to UF, do F' M2 F.
c. Find the edge piece that belongs to the bottom of the Front Face. If it already lies at the bottom of the Front Face and is not upside down, then do F'M'F M' FM'F' to flip it around. If it does not lie in the Middle Layer and is not at the front of the Bottom Layer, then rotate the whole cube around the U/D axis to bring it to the Front Face and do FM2F'.
d. Hold the cube with the destination location of the piece at the bottom of the Front Face and turn M to bring the piece to the right of the Back Face. Then do one of the following:
   1. To move BR to DF, do F'M'F M F'M F
   2. To move RB to DF, do F'M'F2M'F'
e. Repeat a-d for each pair of edges in the U/D layers.

### Phase 3: Solve Middle Layer

a. Turn M to place the centers correctly with respect to the U/D layers.
b. Place the edges correctly by doing one of the following:
   1. To swap FR-BL, FL-BR do M L2R2 M' L2R2
   2. To swap FR-BR, FL-BL do R2 M2 R2 M2 or (R2 M2)2
   3. To cycle FL->FR->BR->FL do R2 M' R2 M
   4. To cycle FR->FL->BR->FR do M' R2 M R2.
c. Orient the edges by doing one of the following:
   1. To flip FR and BR do RM' RM' RM' RM2 RM' RM' RM' R or (RM')3 RM2 (RM')3 R
   2. To flip FL and BR do RM' RM' RM' RM RM' RM' RM' RM or ((RM')3 RM)2
   3. To flip FR, FL, BR, BL do BUB' RM' RM' RM' RM' BU'B' or BUB' (RM')4 BU'B'

There are not as many variations of the corners first algorithm as there were with the layers algorithm, but the speed of this algorithm can be improved in several ways.

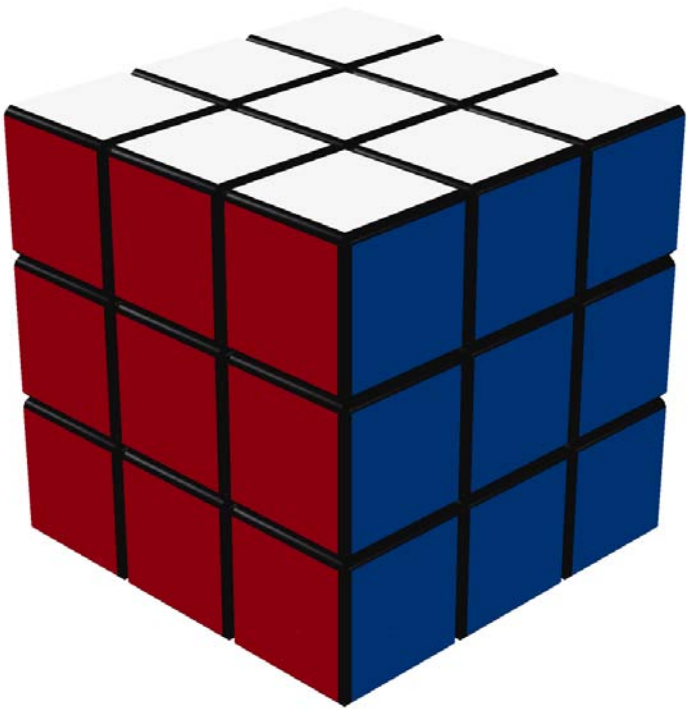
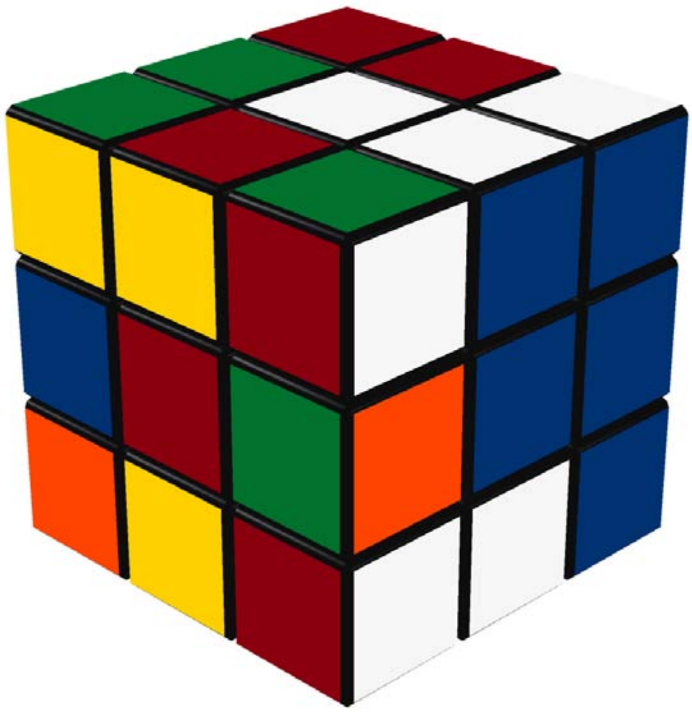
- Phase 2 can be speeded up simply by placing the edge belonging at UF in the FD position first, and when you next place the correct piece at FD, the first piece will be displaced and end up correctly at UF.
- Another variation of phase 2 involves solving the edges one by one instead of in pairs. The basic idea is to rotate the layer opposite the destination layer to bring an unsolved edge to the Front Face. This way you can solve 3 edges in each layer singly without disturbing solved edges. The final two edges are then placed as before.
- Phase 3 can be done quicker by using shorter sequences, or by memorizing sequences that place and flip edges at the same time. There are only 20 sequences needed for this.

## Solution 1 Example

### Scrambling Sequence

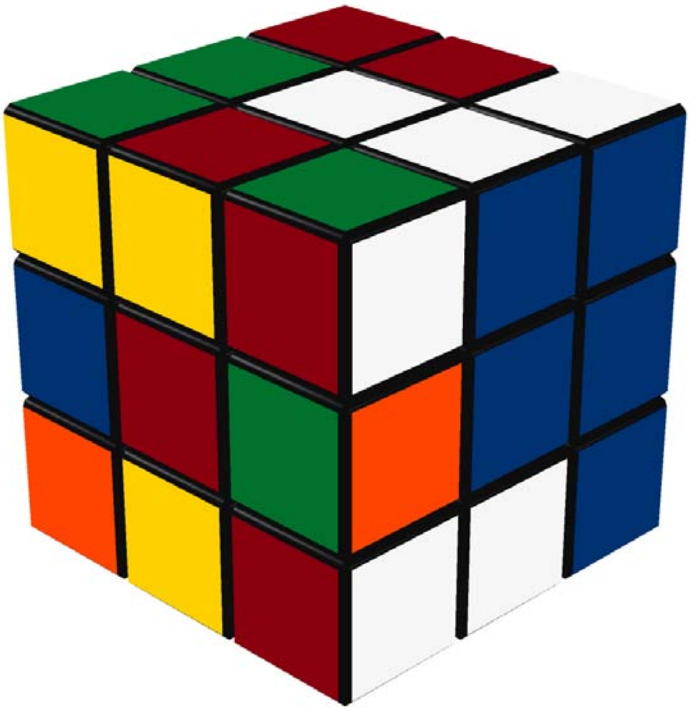
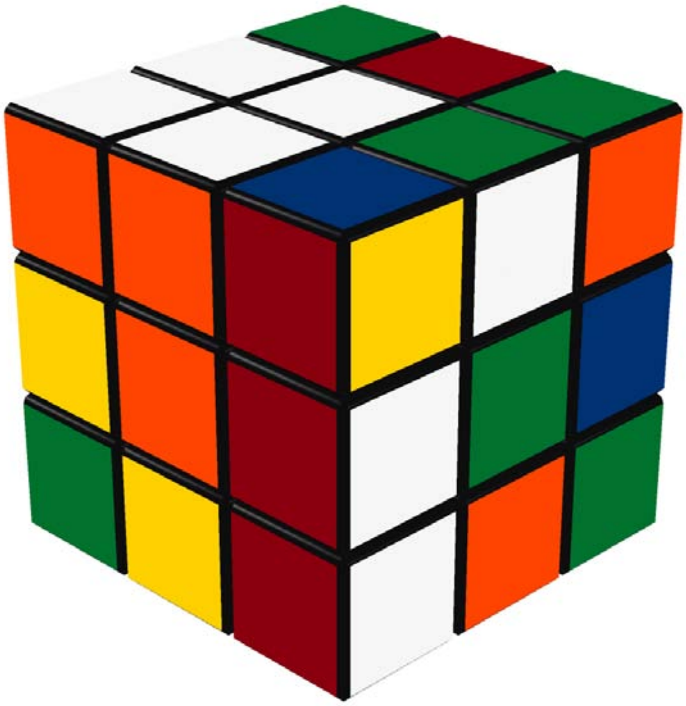
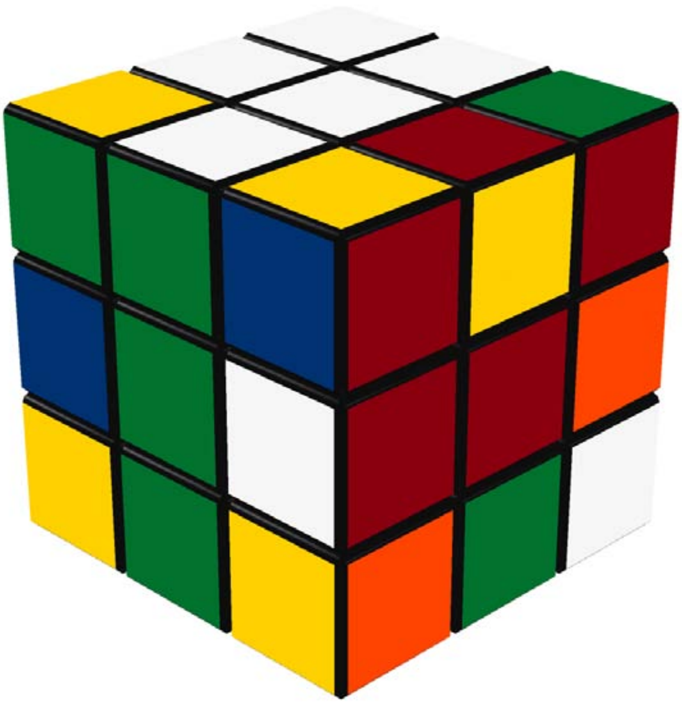
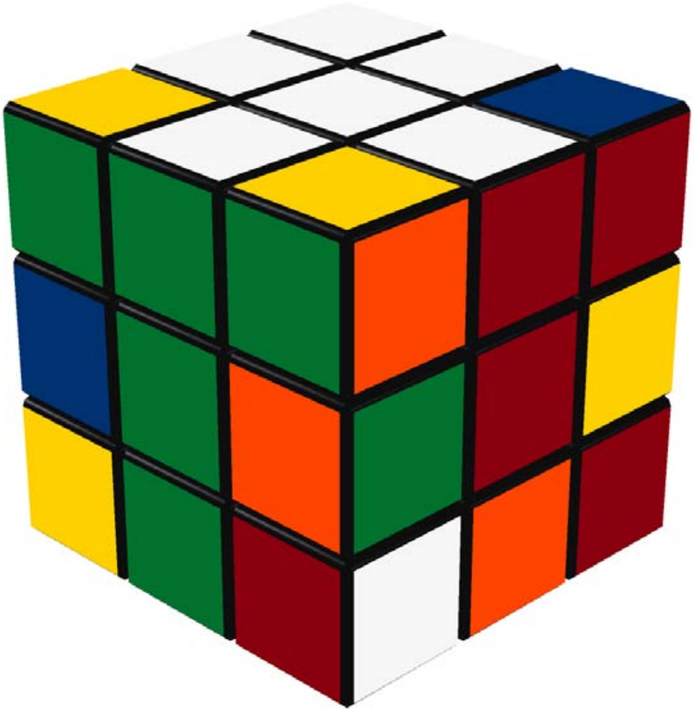
A 25-move scrambling sequence has been chosen for this example:

U B' R2 D' U' R U2 B R' B2 L2 R F2 R2 U2 R B U2 F2 L2 F2 D R B2 R2

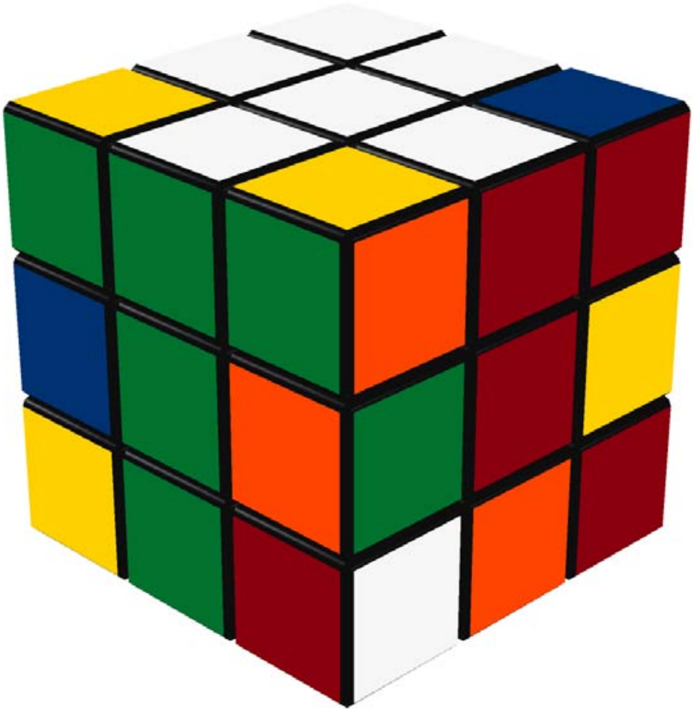
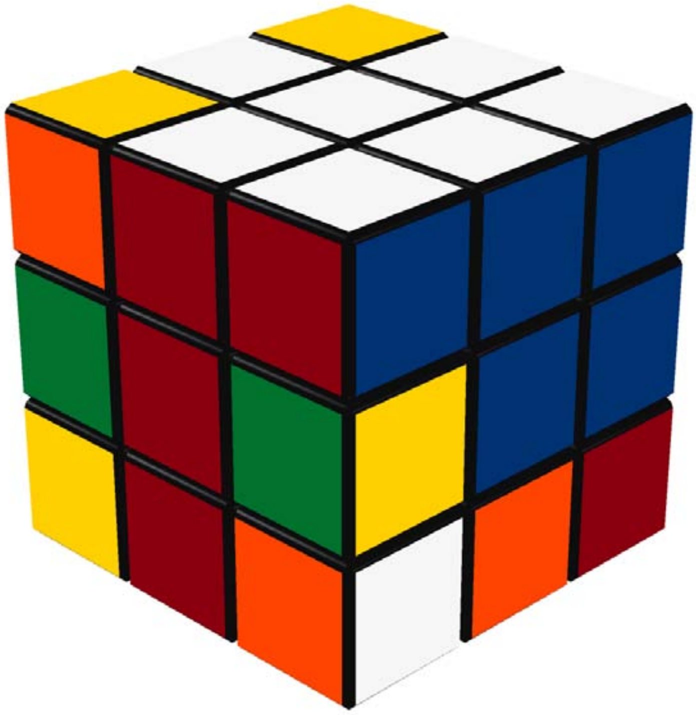
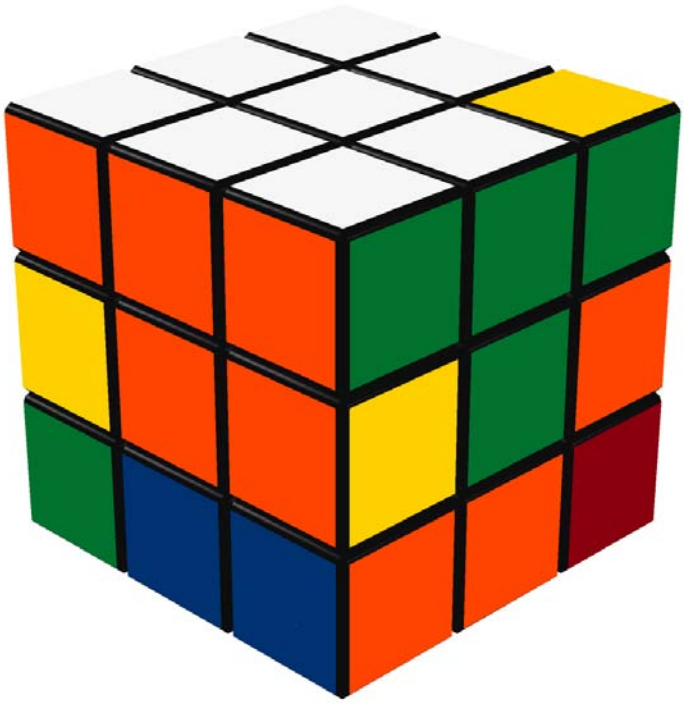
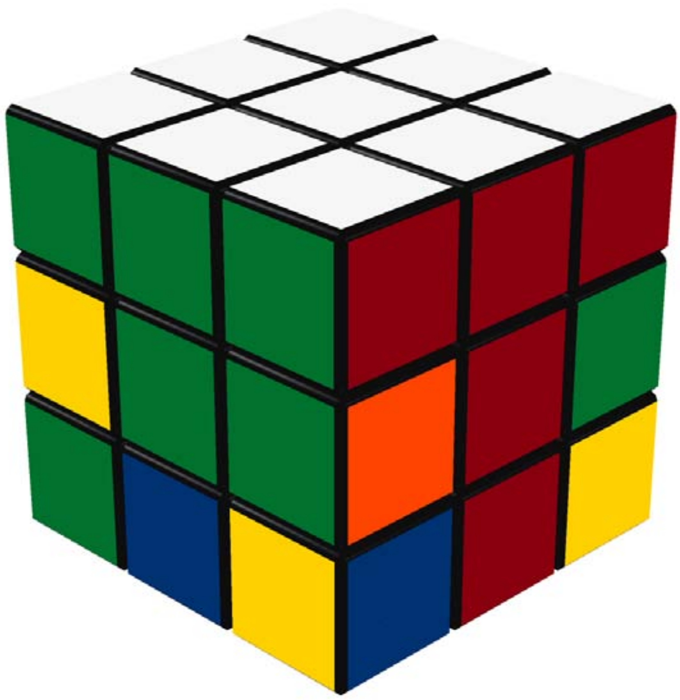
| Scrambled Cube | |
|---|---|
|  |  |
| Initial State (CU' CF) | Scrambled State |
| Scrambling Sequence | |
| CU' CF U B' R2 D' U' R U2 B R' B2 L2 R F2 R2 U2 R B U2 F2 L2 F2 D R B2 R2 | |

## Phase 1: Solving Top Layer Edges

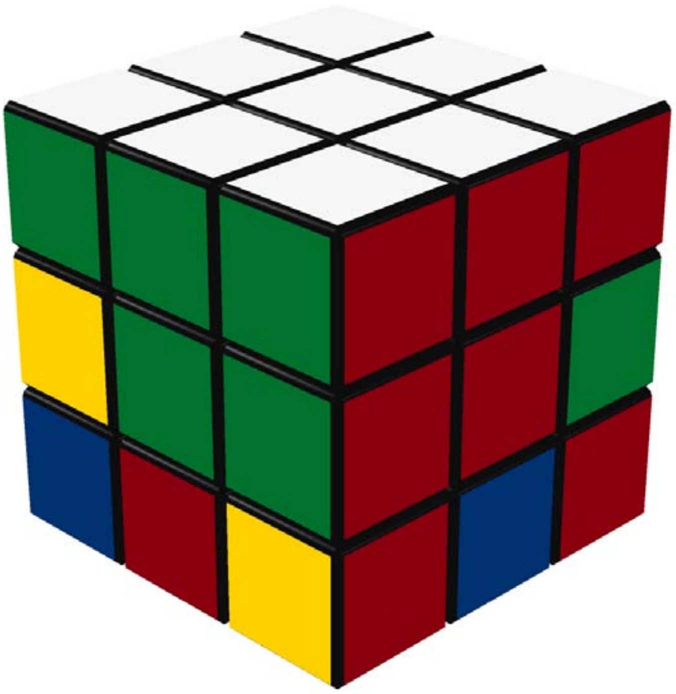
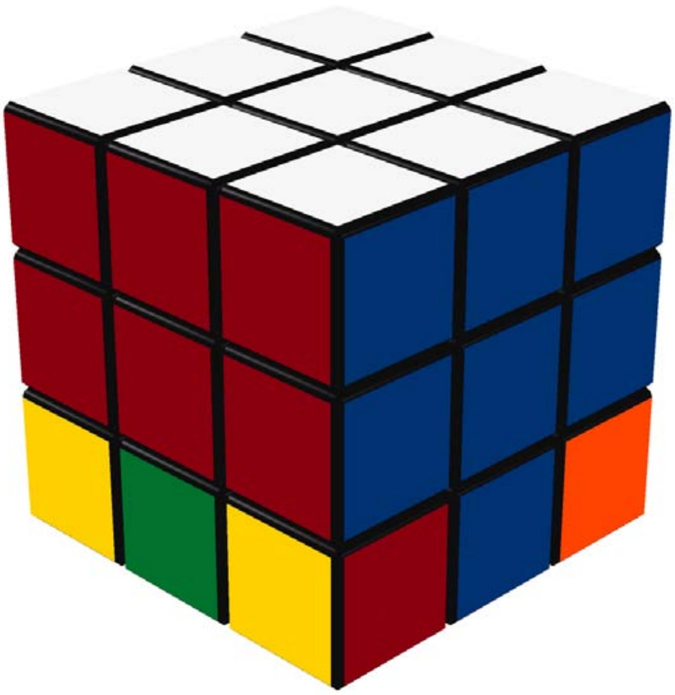
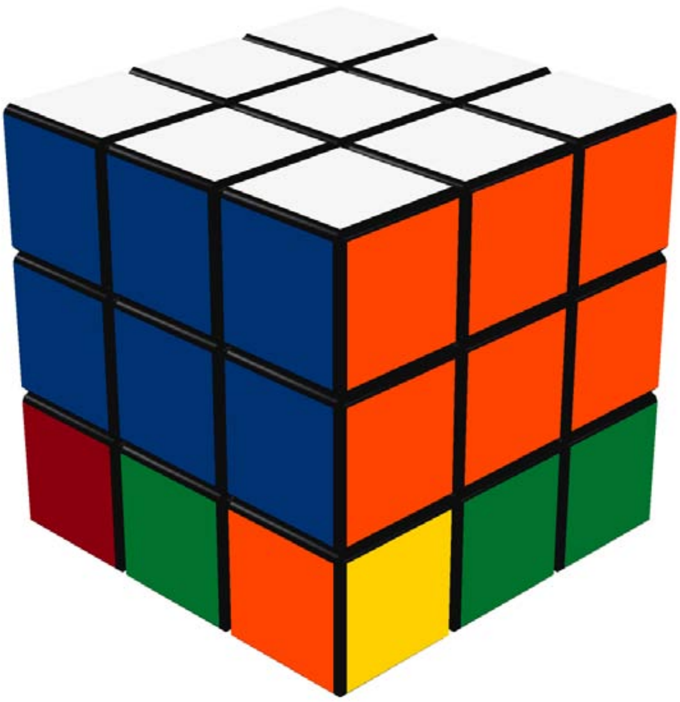
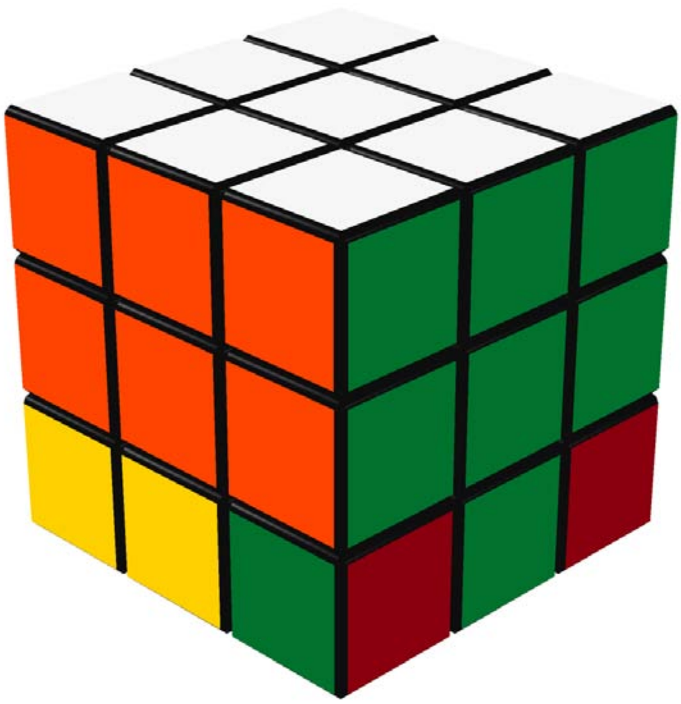
The cube is held with the white center piece on the Top (U) Face. A white cross should first be created on the Top (U) Face. In this example, sections of the white cross are solved in the following order – blue, orange, green, red, as indicated in the Rubiks.com Solution Guide. However, this order may be different if shorter sequences are searched for.

| **Phase 1 – Top Layer Edges 1 & 2** | |
|---|---|
|  |  |
| Scrambled State (White-Blue Edge already solved) | White-Orange Edge Solved |
| | Sequence (Conjugations) |
| | CU2 D (U' (F' R F) U) |
| **Phase 1 – Top Layer Edges 3 & 4** | |
|  |  |
| White-Green Edge solved | White-Red Edge Solved |
| Sequence (Conjugations) | Sequence |
| CU F2 (U' (F' R F) U) | R |

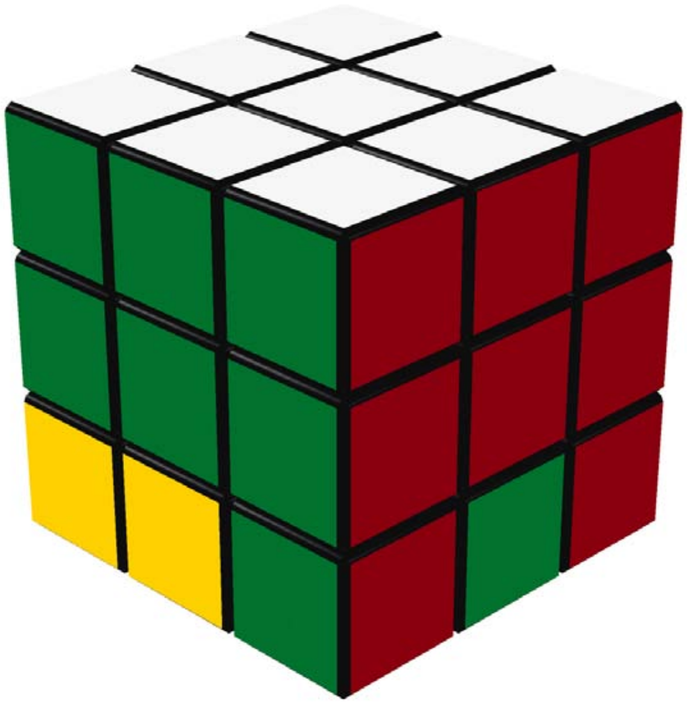
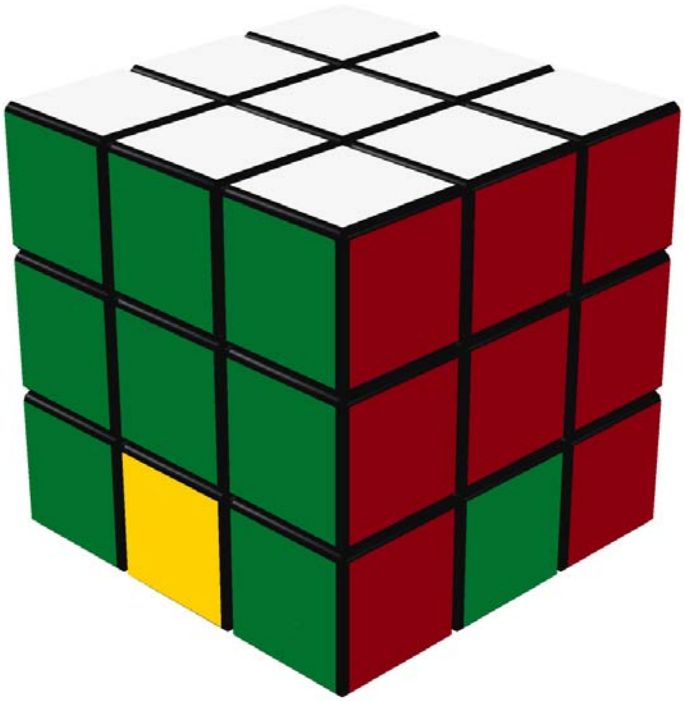
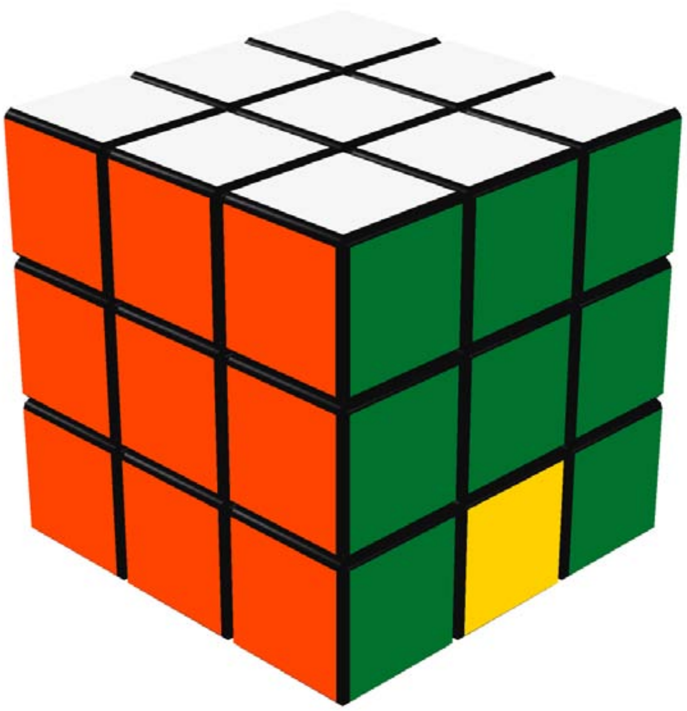
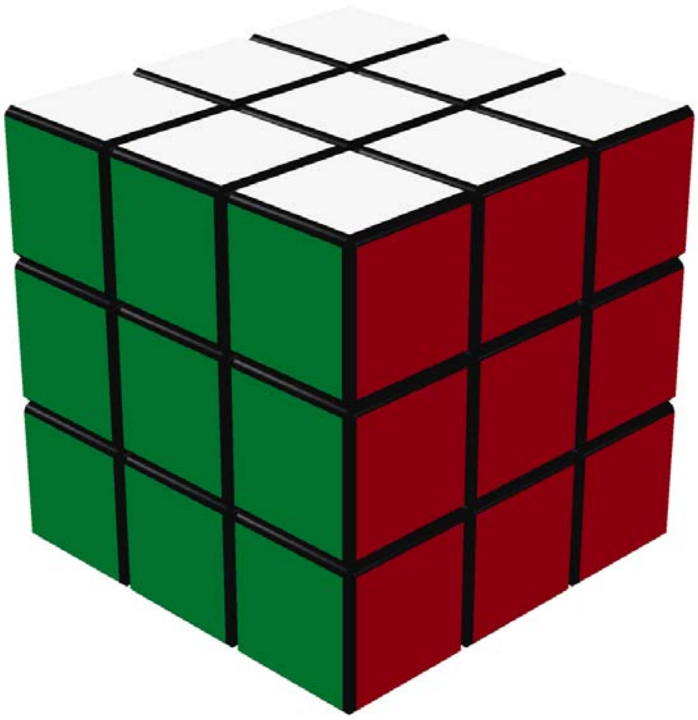
## Phase 2: Solving Top Layer Corners

| **Phase 2 – Top Layer Corners 1 & 2** | |
|---|---|
|  |  |
| White-Blue-Orange Corner already solved | White-Red-Blue Corner solved |
| | Sequence (Conjugation) |
| | CU D (R' D' R) |
| **Phase 2 – Top Layer Corners 3 & 4** | |
|  |  |
| White-Orange-Green Corner solved | White-Green-Red Corner solved |
| Sequence (Conjugation) | Sequence (Conjugations) |
| CU2 D2 (R' D' R) | CU (F D' F') (R' D2 R) |

## Phase 3: Solving Middle Layer Edges

| **Phase 3 – Middle Layer Edges 1 & 2** | |
|---|---|
|  |  |
| Green-Red Edge solved | Red-Blue Edge solved |
| Sequence (Conjugation + Commutation) | Sequence (Conjugation + Commutation) |
| D' (F D' F') (D' R' D R) | CU D2 (F D' F') (D' R' D R) |
| **Phase 3 – Middle Layer Edges 3 & 4** | |
|  |  |
| Blue-Orange Edge solved | Orange-Green Edge solved |
| Sequence (Conjugation + Commutation) | Sequence (Conjugation + Commutation) |
| CU D' (R' D R) (D F D' F') | CU D' (F D' F') (D' R' D R) |

## Phases 4 – 7: Bottom Corners & Edges

| **Phases 4 & 5 – Positioning & Orienting Bottom Corners** | |
|---|---|
|  |  |
| Four Corners Positioned – Two Corners Oriented | Four Corners Oriented |
| Sequence | Sequence (Commutation – Monotwist) |
| CU D | (L' U L F U F') D (L' U L F U F')' D' |
| **Phases 6 & 7 – Positioning & Orienting Bottom Edges** | |
|  |  |
| Three Bottom Edges Cycled | Two Bottom Edges Flipped – Cube Solved ! |
| Sequence (Edge Cycling) | Sequence (Edge Flips) |
| CU' L' R F' L R' D2 L' R F' L R' | CU F U' D R2 U2 D2 L D2 L' D2 U2 R2 D' U F' D2 |

# Rubik's Cube Basics (See Jaap's Puzzle Page)

## Permutations

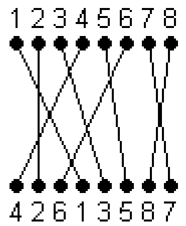
Suppose you have the list of numbers (1,2,3,4,5,6,7,8). A **permutation** of these numbers is simply another list of these numbers in any order, for example (4,2,6,1,3,5,8,7). Every number on the list must be used exactly once. At first sight this is not related to the cube, but suppose you write down a list of a cube corner pieces. Any move on the cube then mixes the corner pieces, and therefore corresponds to a new list with the same corner pieces but in a different order. Any move (or move sequence) is therefore a permutation of the corner pieces. Of course, the same can be said of the twelve edge pieces.

By examining what permutations can do, we can therefore examine how the pieces of the cube move. The numbers in our list which we permute are not really important. What is important is how they move. Usually a permutation uses numbers, but these numbers can represent any item, for example the moving pieces of the cube, or any other objects that can be rearranged. A permutation therefore embodies only the movement of the items.

There are many different ways to write down a permutation. A common way is to write the original list of numbers on one line, and the new list directly below it on another line. For the list of numbers above, we get:

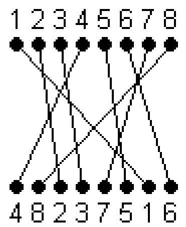
1 2 3 4 5 6 7 8
4 2 6 1 3 5 8 7

To make this more visual, we can make a line diagram by drawing straight lines between numbers of the two lists to show exactly how each item on the list moves.

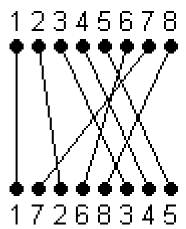



In this example the lines cross 8 times, so we can say that 8 is the permutation **crossing number**.

An important aspect of permutations is that they can be combined. On the cube, for example, one sequence of moves can be followed by another. In other words, the pieces are rearranged in one way, and from this new position they are rearranged in another way. Suppose we combine the permutation above with the following permutation:




Only movements are important, so even though this second permutation is also written using numbers 1 to 8, when we combine them, we only look at the lines, i.e. at what movements the permutation indicates. To combine them, we draw the line diagrams one below the other and follow the lines. By straightening the lines we get:



Let’s look at the crossing numbers of these permutations. The first two have crossing numbers 8 and 15. The combined drawing therefore has 15 + 8 = 23 crosses but when the lines are straightened, we get a crossing number of 11. If you think of the lines as loose threads, and that you physically untangle them, you will see that each time you uncross two threads, the number of crossings always decreases by 2. When you untangle all the threads, the number of crossings then decreases by a multiple of 2. Therefore, although the final permutation's crossing number is not simply the sum, it will have the same parity as the sum (i.e. will also be even or also be odd). Lets call a permutation odd if its crossing number is odd, and even if its crossing number is even. This is the **parity of the permutation**.

It is now easy to see that when combining permutations, their parities follow the same rules as numbers:

odd + odd = even, even + even = even, odd + even = odd, even + odd = odd

On the Rubik's Cube, a single quarter turn of a face is an even permutation. To see this, number the corners of the face 1-4 and the edges 5-8. A quarter turn is then represented by the permutation (2,3,4,1,6,7,8,5) which has crossing number 6 if you draw it. Combining even permutations will always give another even permutation, so only *even* permutations of the pieces of the Rubik's Cube are possible. This shows that it is impossible to swap two pieces (edges or corners) without modifying anything else.

## Disjoint Cycle Notation

A different but very useful notation for permutations is **disjoint cycle notation**. In this notation, the first permutation we used becomes (14)(356)(78). This means that 1 moves to the position that 4 was in, and 4 moves to the position that 1 was in. Piece 3 moves to the position that 5 was in, 5 moves to where 6 was in, and 6 moves to where 3 used to be. Pieces 7 and 8 swap places just like pieces 1 and 4. Piece 2 does not move because it is not mentioned, although you could state so explicitly by including the 2 like this (14)(2)(356)(78).

Each bracketed part of this is called a **cycle**. (14) and (78) is a **2-cycle**, and (356) is a **3-cycle**. These cycles are called **disjoint** because they use different numbers, and so act on *different* pieces. It is easy to see that the parity of a 2-cycle is odd, and that the parity of a 3-cycle is even. The combined permutation (14)(356)(78) is therefore odd + even + odd = even. In general the parity of an n-cycle will be the parity of number (n-1) for any number n.

## The order of a permutation

The **order of a permutation** is the number of times it has to be performed before the pieces are back to their initial positions. This is where cycle notation is very useful.

A 2-cycle is a single swap, so if this is performed twice then the pieces are back where they started. A 2-cycle therefore has order 2. Similarly, a 3-cycle will have order 3, and an n-cycle order n.

Now let’s consider a permutation like (14)(356)(78) which is composed of several cycles. If it is performed twice, or in fact any even number of times then the 2-cycle will disappear. If it is performed three times or any multiple of three, then the 3-cycle will disappear. If it is performed 6 times, which is a multiple of both 2 and 3 then all cycles disappear and this permutation therefore has order 6. In general, the order of a permutation is the lowest common multiple of the lengths of the disjoint cycles.

Let’s see how this applies to the cube. Consider the move sequence FR. If we look at how this sequence moves the pieces, we see that the UFL corner moves to position RUB, corner RUB moves to position RBD, and so on. In cycle notation this permutation is as follows: (UFL,RUB,RBD,RDF,DLF)(UF,RU,RB,RD,RF,DF,LF) This is a 5-cycle of corners and a 7-cycle of edges. Its order is therefore 35, which means that if you constantly repeat the moves FR on the cube, you will have to do it 35 times before the pieces come back to their original positions. If you try this out on a solved cube you will see that the cube is not restored because some corners are twisted. We have only looked at the location of the pieces, and not at their orientation.

## Groups

The collection of all possible permutations of n items form a **group**. A group is simply a collection of things (usually called elements of the group) which satisfy various conditions, which I will list below. We use these conditions implicitly whenever we use permutations, so it is best to state them explicitly now.

1. Any two elements of the group can be combined, and this results in another element of the group. As you have seen, two permutations can be combined by performing one after the other, and this will always result in a permutation. If P and Q are permutations, then PQ will mean the permutation resulting from performing first P and then Q. Combining two elements of a group is usually called multiplication of the two elements.
2. There is an **identity**, i.e. an element I in the group such that for any element P in the group we have PI = IP = P. In a permutation group the element I is simply the permutation that does not move anything.
3. Every element has an **inverse**, i.e. if P is an element of the group then there is an element Q in the group such that PQ = QP = I. If you mirror the line diagram of a permutation vertically, you get the line diagram of its inverse. A permutation in cycle notation can be inverted just by writing each cycle in reverse. The inverse of permutation P is denoted by $P^{-1}$, or by P'. On puzzles, you can do the inverse of a move sequence just by undoing the moves in reverse order, i.e. taking back the moves you did.
4. The multiplication is **associative**, i.e. if P, Q and R are elements of the group, then (PQ)R=P(QR). With permutations this is obviously true.

If a group is **commutative**, i.e. we have PQ = QP for any P, Q in the group then it is called an Abelian group. This phenomenon occurs in the Lights Out and Rubik's Clock puzzles, and makes these much simpler to solve because the order in which the moves are performed does not matter. Generally, permutation groups occurring in puzzles are not Abelian.

All possible movements of the pieces on the Rubik's cube also form a group, the **Cube Group**. At first sight this is not a simple permutation group because the orientation of the pieces matters, but you could consider it a permutation group of the 48 moving facelets instead of the 20 moving pieces. On a pedantic note, it is technically incorrect to say that the positions of the Rubik's cube form a group. A position is reached from the solved cube by moving the pieces in some way, and it is those movements that form the group because movements can be combined. On the standard Rubik's Cube this is not a very important distinction, but on other puzzles like the 4x4x4 Cube it is. This puzzle has center pieces that look the same, and so there are positions which cannot be distinguished from others. Thus different permutations seem to correspond to the same position, or permutations which seem to do nothing in one position will change things in another. The permutations still form a group, but the positions do not (unless you mark the centre pieces so they can be distinguished).

## Conjugation

If P and Q are elements of a group, then the **conjugate** of Q by P is the element PQP'. This is one of the most useful concepts for solving a puzzle like the Rubik’s Cube. Let’s illustrate this with an example on the Rubik's Cube. Suppose you know that the move sequence Q = R' L F2 R L' U2 cycles three edges around, viz. (UB,UF,DF) but that you also want to cycle 3 more edges of the cube, for example (UR,UF,UL). We would like to know how to cycle them if they were placed at UB, UF and DF, so we simply put them there, for example by doing the sequence P = F2U. This sequence moves other pieces as well, but that does not matter. So after we placed them in position with P, cycled them with Q, we now move everything back in place by doing P'. The relevant edges have been cycled as we wanted and any other pieces that were moved by P are put back in place by P'. Therefore PQP' = F2 U R' L F2 R L' U F2 cycles just the three edges (UR,UF,UL).

As you can see from this example, if you have a sequence that performs a certain task on particular pieces of the puzzle, conjugation will also allow you to perform the same task on any other similar pieces of the puzzle. So, if you can flip two edge pieces then you can flip any two of them, or if you can twist two corners then you can twist any two of them, and so on.

## Commutation

Conjugation allows you to apply a specific sequence more generally, but you still need to find that specific sequence to begin with. This is where **commutation** is useful. If P and Q are elements of a group, then PQP'Q' is called a **commutator**. If P and Q do commute (for example if they are disjoint, like moves R and L on the cube) then PQP'Q' = QPP'Q = QIQ' = QQ' =I. The commutator can be seen as an indication of whether P and Q commute, and by how much. If P and Q are *nearly* disjoint, then the commutator will move a *very few* pieces, and it therefore often performs a useful function when solving a puzzle.

The simplest commutator on the cube uses single face moves for P and Q, for example FR'F'R. This cycles three edges (FU,FR,UR), and two pairs of corners (UFL,BRU) and (URF,RDF). Note that corner UFL moves to BRU, which in turn moves to LUF. This is twisted anti-clockwise compared to the original position FLU, so if we perform this cycle twice, these two pieces will be back to their original positions but will both be twisted anti-clockwise. The other corner 2-cycle of FR'F'R twists clockwise. We could adapt cycle notation to show this as follows:

FR'F'R = (UFL,BRU)- (URF,RDF)+ (FU,FR,UR)

Doing this twice, we get:

$(FR'F'R)^2$ = (UFL)- (UBR)- (URF)+ (DFR)+ (FU,UR,FR)

Doing this three times, we get:

$(FR'F'R)^3$ = (UFL,UBR) (URF,DFR)

Theoretically these moves and their conjugates are enough to perform any even permutation on the corners, and any even permutation on the edges. Any single quarter turn of a face is an odd permutation on both corners and edges, so just using what we have so far we could position all the pieces of the Cube. What remains is just to orient them.

Commutation works best when P and Q are *nearly* disjoint.

Therefore let's choose Q to be a turn of the U face, and P so that it affects only a *single* piece in the U face. An extremely useful choice is the **monotwist** P = R'DRFDF'. This twists one corner (URF)+ and does not affect anything else in the U layer. The bottom half of the cube is messed up but that does not matter. We now have the following very useful sequences:

PUP'U' = R'DRFDF' U FD'F'R'D'R U' = (URF)+ (UBR)-

PU2P'U2 = R'DRFDF' U2 FD'F'R'D'R U2 = (URF)+ (ULB)-

PU'P'U = R'DRFDF' U' FD'F'R'D'R U = (URF)+ (UFL)-

You can now twist any *two* corners on the cube.

Another good choice for P is the **monoflip** P = FUD'L2U2D2RU = (FU)+, which will allows you to flip any edge on the cube, and another one is P = R'DR which gives you a simple 3-cycle of corners.

It is also productive to let Q be a move of a Middle Slice, for example MR (if you look squarely at the R face of the cube, MR is a clockwise quarter turn of the middle slice just behind the R layer). If we let P = F2, which is a **monoswap** of edges (DF,UF) and Q = MR, then we get a 3-cycle of edges PQP'Q' = F2MRF2MR' = (DB,DF,UF).
If P = F2 and Q = MR2, we get the 2-H pattern PQP'Q' = F2MR2F2MR2 = (DF,UF) (DB,UB).
If P = MF and Q = MR, we get the 6-spot pattern PQP'Q' = MFMRMF'MR' (swap of 6 *adjacent* centers).
If P = MF2 and Q = MR, we get the 4-spot pattern PQP'Q' = MF2MRMF2MR' (swap of 4 *opposed* centers).
Another good choice is P = FU'RF'U, a neat **monoflip** (FU)+.

## Size of the group

You may have noticed that when you use a commutator for twisting corners, you will always twist two corners in opposite directions. Commutators can also flip only pairs of edges. It turns out that this is enough to solve the cube because it is impossible for a single piece to be turned without turning other pieces.

Every corner piece has one facelet that belongs in the U or the D face. For a corner which has been moved anywhere on the cube, lets define its twist as follows:

- Its twist is 0 if its U/D facelet is in the U or D face.
- Its twist is +1 if the piece has been turned *clockwise* from the 0 twist orientation.
- Its twist is -1 if the piece has been turned *anti-clockwise* from the 0 twist orientation.

Twisting a corner clockwise will increase its twist by one. We have to work modulo 3 however, because 3 twists is the same as no twist at all, and so a twist value of +2 is really only a twist of + 2 - 3 = - 1.
Similarly anti-clockwise twisting decreases the twist by 1 modulo 3 (i.e. a twist of -2 is just a twist of +1).
If you turn the U or the D face, the twist of the corner pieces do not change. If you turn any other face by a quarter turn, then the twist of two of the corner pieces increases, and the twist of the other two corners decreases. In any case the total twist of all the corners does not change modulo 3. In the starting position the

cube has a total twist of 0, and therefore this remains equal to 0 however mixed up the cube can be. This shows that it is impossible to twist a single corner in isolation, and if you twist only 2 corners then they must go in opposite (twist) directions.

A very similar method can be used for edges: define the flip of an edge as 0 or 1 (modulo 2) and show that the total flip remains 0 for any move performed, this means that no edge can be flipped in isolation. Another way is to look at the permutations of edge facelets. A quarter turn of a face is an even permutation of edge facelets (two 4-cycles) so any move sequence will give only even permutations of edge facelets. A single edge flip is an odd permutation of edge facelets and hence this is not possible without taking the cube apart.

In all we have now seen three restrictions on the possible arrangements of the pieces of the Rubik's Cube. The total corner twist must be zero modulo 3, the total flip of edges must be zero modulo 2, and the parity of the piece permutation must be even. If you were to take the cube apart and randomly put it back together again, there would be a 1 in 3 chance of having the right corner twist (since all three possible twist values are likely equal). Similarly there is a 1 in 2 chance to get the total edge flip correct, and a 1 in 2 chance of getting the right permutation parity. Putting this together, we find there is a 1 in 12 chance that a randomly assembled cube is solvable.

In a later section on Counting we shall count how many randomly assembled cubes there are. The actual number of cube positions is then one twelfth of that.

## Subgroups

The Rubik's Cube Group is **generated** by the moves {F, B, R, L, U, D}, because by definition any movement of the pieces is done by moving the faces one at a time. Suppose instead that from a solved cube you only do half turns, i.e. you only use the moves {F2, B2, R2, L2, U2, D2}. Clearly this does not generate the whole Cube Group; for a start each face of the cube never has more than two colors. Nevertheless, the permutations these moves generate do form a group because any two permutations made from half turns will combine to give another permutation made from half turns. This group is usually called the **Square Group**.

The Square Group is called a **subgroup** of the Cube Group, because it is a subset of it and is a group in its own right. Of the four conditions listed earlier that define a group, we only need to check the first one – that combining two permutations of the square group also gives a position that can be solved using only square moves. The other three conditions are inherited automatically from the full Cube Group.

There are many pretty patterns in the Square Group, and no doubt you already know some of them. The 4-spot pattern for example can be made by R2L2U2R2B2R2L2F2L2U2, which is 10 half turns. It can be reached in fewer turns if you allow the cube to temporarily leave the square group by using quarter turns. The 4-spot can be reached in 8 turns by R2L2UD'F2B2UD'.

There are other interesting subgroups of the Cube Group, for example the **Slice Group** (generated by all slice moves: FB', RL', UD'), the **Antislice Group** (generated by all antislice moves: FB, RL, UD), and many others.
It is a strange fact that you only need 5 faces to solve the cube. In other words, {F, B, R, L, U} generates the whole Cube Group. If you know how to solve the cube layer by layer, it is fairly easy to do it without turning the first layer at all. It is therefore fairly obvious that you can solve the position with just the first layer rotated without using any turns of that layer. So one face can be turned by using only the other five. A simple way to prove it is by means of the sequence P = R2L2U2R2B2R2L2F2L2U2, which does the 4-spot. It moves all the pieces from the D layer up to the U layer in the same relative positions without actually turning the D layer, so that PUP' has the same effect as D. The same reasoning shows that the whole Square Group is also generated by just 5 faces.

Not all subgroups are generated by a restricted set of cube moves. In fact most subgroups are better described by the effects of its permutations on the cube. For example, consider all permutations that do *not* move the UFR corner. It is fairly easy to show that these form a group – any combination of such permutations still doesn't move that corner. A very often used subgroup is the **U-Group** – the set of all permutations that only move pieces in the U layer, and leave the bottom two layers intact. A lot of research has been done to find short move sequences for these permutations so that solving the last layer can be done as fast as possible.

## Cube Group Center

One particular subgroup that is fascinating is the **Center** of the Cube Group. The Center of a Group includes all elements which commute with everything, i.e. if C lies in the Center, then CP = PC for all elements P in the Group. Since CP = PC means that PCP' = C, the element C does not change under any conjugation. The only way for this to be true is if it does not move any piece from its place. Also, if C twists one corner, then by conjugation it must twist all corners in the same direction which is not possible on the cube. Similarly, if it flips one edge then it must flip all of them, but this is possible on the cube. The only elements in the center are therefore the identity I and the **Superflip** which flips *all* 12 edges.

The Superflip can be done by the sequence: ((MR U)4 CR CD)3. This rather cryptic sequence will need a little explanation. The notation MR is used to denote a move of the middle slice adjacent to the R face in clockwise direction if you look at it from the R side. You have to do MRU four times. Now you have to do CRCD which are rotations of the whole cube in the same directions as the moves R and D (these two cube rotations together are the equivalent of a rotation about the UFL corner). Now you have to repeat everything you have done so far twice more (3 times all together).

The fact that the Superflip commutes with everything can be used for some impressive tricks. Suppose you apparently mix the cube up, but actually perform the Superflip (S). You hand the cube to someone and ask them to do one or two random moves (P) and then hand it to you behind your back. You have not seen which moves he did, nor which way up the cube is handed to you, and yet you seem to solve it behind your back. Simply perform the Superflip again, and bring it out to the front while you say that you are nearly finished. The cube will only need one or two moves to be solved which can be done instantly by sight.

This works because SPS = SSP = IP = P so you only need to undo the few random moves that the cube was given.

I explained the above trick to the wonderfully clever magician Jerry Sadowitz, and he very quickly came up with the following idea based on this. You show two Rubik's Cubes, both of which are mixed up differently. You ask someone to mix the cubes further, as long as every move that is performed on one is also performed on the other. You do not see these moves, and yet when you are handed either one of these cubes behind your back, then without even seeing the other cube you can mix yours so that it is the same as the other one. You should be able to figure out how this works now.

## Supergroup

On the Rubik's Cube the center pieces of each face do not change position, but they do rotate. On a normal cube this is not visible, but there are cubes with pictures or other designs on them where the orientation of the face centers does matter. Each quarter turn is an odd permutation of the corners of the cube (and also odd of the edges), so it is impossible for a face center to move a quarter turn without moving any other pieces at the same time. It is possible however for a single center to do a half turn in isolation, or for two centers to do a quarter turn without moving other pieces.

The group of all permutations of a cube with *visible* centers is called the **Supergroup**. It is $4^6/2 = 2048$ times as large as the normal Cube Group.

We already saw that $(FR)^{35}$ leaves the pieces in the same position, but twists some corners. Doing this three times, i.e. $(FR)^{105}$, will therefore not move any pieces. It does however turn the F and R face centers 105 times, so it leaves them both twisted by a quarter turn. Many other move sequences have an order (in the normal Cube Group) that is not a multiple of 4, and they will usually cause centers to be twisted when repeated. These methods are not really practical however.

By using commutators it is not very hard to find shorter move sequences for twisting face centers. For example let P be the 6-spot pattern (e.g. MRMFMR'MF') and Q a face turn (e.g. U). The result is then a sequence which turns one face center clockwise, and an adjacent face center in the opposite direction. The sequence RL'FB'UD'R'U'DF'BR'LU which turns centers U and R' is derived that way. For opposite faces you could use the 4-spot instead, and this gives RL'F2B2RL'URL'F2B2RL'D' which turns U and D' centers. A short sequence for a half turn of a face center is more difficult to find. Let P be the sequence RLU2R'L'U2. This swaps one pair of opposite corners and one pair of opposite edges in the U layer. Commuting it with U therefore gives you a sequence which has the same effect as U2 on the corners and edges, but which does not move the U center. Follow it with U2 and you have only moved the U center. Thus $(RLU2R'L'U)^2$ will move the center U2.

## Metrics

It would be nice if we could measure how far away a cube is from being solved. If we could do this perfectly, we would be able to solve the cube in the quickest possible way simply by doing any move that brings it closer to the solved position. This quickest way is usually called God's Algorithm. It is of course not so simple to measure distances on the cube. In mathematics, a measure of some kind of distance between two points is called a metric.

A **metric** D is a function which should satisfy a few fairly obvious conditions:

1. D(a,b)>=0, or any distance is positive or 0.
2. D(a,b)=D(b,a), or the distance from a to b is the same as from b to a.
3. D(a,b)=0 if and only if b=a, or distances between differing points are always non-zero, and the distance from one point to itself is always zero.
4. D(a,b)+D(b,c)>=D(a,c), also called the triangle inequality, means that if you go from a to c, the distance can only get longer if you detour via b.

On a cube, a reasonable metric would be the minimal number of moves to get from one position to the other. You can check that the conditions above are true, provided that the inverse of a single move is also considered a single move.

There are however several different opinions as to what constitutes a single move. Some people prefer the Quarter Turn Metric (QTM), i.e. only a quarter turn (in either direction) of a face is considered a single move, while other people use the Half Turn Metric (HTM) in which half turns are also considered to be a single move (this is also called the Face Turn Metric, and is often denoted by q + h). For example the Pons Asinorum or the 6X pattern reached by the sequence U2D2F2B2R2L2 is 12 moves from start in QTM but 6 moves in HTM. Some people even consider a slice move as a single move, and in this Slice Move Metric (SMM) the Pons would be only 3 moves from start.

The example of the 6X pattern above works only because it is known that it cannot be reached by a shorter route in any of the metrics used here. To prove this for the QTM, consider that each of the 12 edges needs 4 quarter turns to get into position, each move only moves 4 edges, and so you need at least 12 moves to reach/solve the 6X position.

Generally if you look at a move sequence, counting the number of moves in the sequence will only give you an upper bound for its distance from start. There may well be a shorter, more direct sequence. You can even prove this obvious result by using the triangle inequality.

**God's Algorithm**

God's Algorithm of a puzzle is the solving algorithm that solves it in the fewest number of moves from any position. In other words, from any position of the puzzle the algorithm gives a move that brings it closer to the starting position. Note that closeness is measured by a metric, which depends on exactly what is considered to be a single move. At the moment this algorithm is not known for the Rubik's Cube, but for several smaller puzzles like the Skewb, the Pyraminx, the Diamond, and the 2x2x2 Cube it is known.

It is usually calculated by computer as follows:

1. Set up a large array with entries for every possible position of the puzzle.
2. Mark the solved position (or positions) as being at distance 0.
3. If all positions of distance <=L are known, then those at distance L+1 can be found like this: For each position at distance L, try all possible moves. If an unknown position is reached, then that must be at distance L+1.
4. Repeat step 3 until all positions have known distance.

This method requires a lot of memory, at least 2 bits for each possible position, so it is not practical for the Rubik's Cube and is not likely to be in the near future. See Computer Puzzling for more details on how to calculate this in general.

We can however find some simple lower bounds for God's Algorithm for the Rubik's Cube. Suppose we begin from a solved cube, and try to count how many positions that can be reached in a certain number of moves (HTM). There are $6 \cdot 3=18$ choices for the first move. We can assume we don't move the same face twice in succession, so there are $5 \cdot 3=15$ choices for the second move, and the same for each subsequent one. The total number of positions that can be reached in n moves is therefore at most:

$$1+18+18 \cdot 15+18 \cdot 15^2+18 \cdot 15^3+....+18 \cdot 15^{n-1}$$

This is of course an overestimate, because many positions can be reached in several ways. This number first exceeds the total number of positions the cube has when n = 17, so there are positions which need at least 17 moves to solve. A better counting argument that takes into account that RL = LR etc., gives n = 18.
If we use the QTM, then the same argument gives as an upper bound on the number of positions reachable after n moves:

$$1+12+12\cdot 11+12\cdot 11^2+12\cdot 11^3+....+12\cdot 11^{n-1}$$

Here we get n = 19, but this can be much improved because it does not take into account that no more than 2 consecutive quarter turns of the same face is ever done. The best calculations show that some positions require at least 21 quarter turns.

These lower bounds have been improved upon, mostly by computer searches, by showing that certain positions need more moves to be solved. The superflip for example has been shown to need at least 20 moves in HTM, or at least 24 moves in QTM. The position reached by combining the 4-spot pattern and the superflip needs at least 26 moves in QTM.

Every algorithm that can solve the cube will take longer than God's algorithm, and therefore we have upper bounds for God's algorithm too. The best known solving algorithms take 28 moves in HTM or 36 moves in QTM. These methods involve large computer databases and are not practical for humans.

Thus God's algorithm will have a maximum length between 26 and 36 in QTM, or 20 and 28 in HTM.

## Counting

It often happens that you want to find out how many positions a particular puzzle has. This can be tricky, but the difficulty mainly lies in determining the restrictions, such as parity restrictions or twist restrictions that we saw earlier. Apart from those, it is possible to count the number of positions is of a puzzle by imagining it taken apart, and counting the number of possible ways to put it together. To do this, we need to count the number of orientations, permutations, and combinations.

## The number of orientations

Suppose you are assembling a Rubik's cube, and are inserting a corner piece. Wherever you put it, it has three possible orientations. You have this choice of three orientations for each of the eight pieces, and the total number of possibilities is multiplied by 3 for every corner you place. The total number of orientations for the corners is therefore $3^8$ = 6561. As we have seen before there is a constraint (the total twist should be zero), so for the resulting position to be solvable we need the orientation of the last corner to be consistent with that of the other seven. There are therefore really only $3^7$ = 2187 possible corner orientations.

The number of edge orientations is calculated in much the same way. There are twelve edges, each with 2 possible orientations, so at first sight there must be $2^{12}$ = 4096 possibilities. Again there is a constraint so that the orientation of the last placed piece is dependent on that of the others in solvable cubes, so there are really only $2^{11}$ = 2048 edge orientations.

Some puzzles do intrinsically have a constraint on the orientations, but also have some pieces of which the orientation is not visible. The best known example is the Pyramorphix. It is essentially a 2×2×2 cube, and half the pieces have no visible orientation. In this case you can still count the orientations of the normal pieces (giving $3^4$ orientations). The zero-twist constraint has no effect however, since you can simply imagine putting one of the monochrome pieces in last when you are assembling it so that it will automatically be solvable whichever way that piece is oriented. Generally, any puzzle where there is at least one piece without visible orientation will effectively have no constraint on the orientations of that type of piece.

## The number of permutations

Again, let's suppose you are assembling a Rubik's cube, and are inserting all twelve edge pieces one by one. For the first edge there are 12 possible places to put it, for the next there are 11 possibilities left, for the third only 10, and so on until the last edge which goes into the last remaining empty spot. The total number of arrangements is therefore 12·11·10·...·2·1, also called 12 factorial, and which is usually written as 12!.

The corner permutations can be counted in the same way, so there are 8! of them. In general, there are n! ways of permuting of n items.

It is occasionally useful to know how many ways there are to place r pieces amongst n places where r<n. For example, how many ways are there for the U layer edges to be arranged on a mixed cube? This is calculated in the same way as before, except that you stop after the four pieces have been placed. In this case you get 12·11·10·9, which can also be written as 12!/8!. On scientific calculators this is denoted as nPr, which means n!/(n-r)!.

Once again on the Rubik's Cube there is a constraint, due to the permutation parity. The parity of the permutation of all 20 pieces (corners and edges) must be even for the cube position to be solvable. This means that the last two pieces you place (whether they are two edges or two corners) must be positioned in such a way as to make the permutation of even parity. This is possible because swapping them changes the parity from odd to even and vice versa. Therefore the actual number of solvable permutations is half the total number of permutations.

We can now get the number of solvable cube positions by multiplying together all these numbers:

$$3^7 \cdot 2^{11} \cdot 8! \cdot 12!/2 = 43,252,003,274,489,856,000$$

This is therefore also the size of the Cube Group. The total number of possible positions attainable by taking apart and re-assembling it is 12 times this number. From the discussion about conjugation and commutation you can prove that all of these positions can indeed be reached (and solved).

## The number of combinations

There are some puzzles which have identical pieces, for example the centre pieces of the 4×4×4 cube. Swapping centre pieces of the same color is possible, and this does not change what the puzzle looks like. Suppose you gave the four centers of one face markings to distinguish them. These four pieces can be rearranged amongst themselves in 4! distinct ways, but if you remove the markings, all of these positions are identical. The same can be done for each of the six faces. The total number of ways the 24 centre pieces can be arranged is therefore $24!/4!^6$.

We can come to the same conclusion in a different way too, by counting each color separately. How many ways are there for the U face centers to be arranged on a mixed cube? If they were all distinct then there would be 24!/20! = 24·23·22·21 ways to arrange them. Since they look the same, we must divide by 4! to get 24!/(20!·4!). This is the number of ways to place 4 identical items amongst 24 locations, or alternatively the number of ways of choosing 4 locations out of 24. It is often called '24 choose 4'.

The general formula for 'n choose r' is n!/(r!·(n-r)!), and on scientific calculators it is denoted nCr.

The total number of 4×4×4 centre piece arrangements can now be calculated by doing the centres one set at a time. The first four have 24!/(20!·4!) arrangements. The next four can be placed at any of the remaining 20 places in 20!/(16!·4!) ways, the four after that in 16!/(12!·4!), and so on. Multiplying all these together and simplifying gives the same answer as before, namely $24!/4!^6$.

If there are identical pieces, then any permutation parity constraints on those pieces have no effect, at least not on the number of positions. This is simply because swapping two identical pieces would change the parity without changing the position. It is in fact precisely this effect which can cause confusion when solving such a puzzle – to swap two distinct pieces you will have to swap two identical pieces as well if there is a parity constraint.

## Burnside's Lemma

Burnside's Lemma is also called the Pólya-Burnside Lemma, the Cauchy-Frobenius Lemma, or even "the lemma that is not Burnside's". It can be used for counting the number of positions of a puzzle when some of them are considered to be equal due to the fact that the puzzle is symmetric in some way. I have hardly used this on my pages at all, though in some cases I could and should have. Let me first give a simple example where it can be used.

Suppose you have a square, and each of the sides is colored either red or blue. How many of such distinct squares are there? Here it is easy to just count them; all sides the same color (red or blue), 3 sides of one color and 1 of the other (again two possibilities), 2 sides of one color and 2 of the other (adjacent or opposing). There are therefore 6 such squares.

In more difficult cases there are too many to just count them like that. You could then calculate how many there might be (here we would get 16 because each of the 4 sides has 2 possibilities, red or blue), and then see which of these are symmetric and how many times too often they have been counted. Burnside's Lemma actually does something like this.

The lemma states in mathematical terms that if G is a finite group acting on a finite set X, then the number of orbits under this action is given by taking the average number of the fixed points. In our example, the set X is simply all possible squares ignoring symmetries (i.e. all 16), the group G is all different ways we can rotate/reflect the squares, and each orbit is a complete set of squares which are the same under rotation/reflection. The number of orbits, which is what the Lemma calculates, is therefore the number of distinct squares when rotation/reflection is taken into account.

Let's calculate it now. First we must find all the symmetries of the square, i.e. all ways of rotating or reflecting it. These are:

A. Rotation through 90 degrees.
B. Rotation through 180 degrees.
C. Rotation through 270 degrees.
D. Reflection through the horizontal midline.
E. Reflection through the vertical midline.
F. Reflection through one diagonal.
G. Reflection through the other diagonal.
H. The identity.

We must include the identity (rotation through 0 degrees if you like) because we need to use the whole group of symmetries, and groups always have an identity element. Note that the pairs A and C are similar, as are D and E, and also F and G.

Now we have to count the number of 'fixed points' that each symmetry has, in other words how many of the 16 possible squares remain unchanged by each symmetry. A square can only remain the same under a 90 degree rotation if all its sides have the same color. Therefore symmetry A has exactly two 'fixed points', and so does symmetry C.

Symmetry B has 4 'fixed points', because we must have opposite sides of the square the same color, and each pair of opposite sides can be either color. Note that this also includes the squares of one color that we had with symmetry A and C. Symmetries F and G each also have 4 'fixed points', D and E have 8, and finally H has 16 because all squares remain the same if you do nothing to it.

The average of these 8 numbers is therefore (2+4+2+8+8+4+4+16)/8 = 6, as we expected.

Let's now try to apply the Lemma to a real puzzle. It could be applied to count the number of shapes of the Pyramorphix, but they are much easier to count by hand. Instead, lets count the number of distinct patterns on an Orbix puzzle. This is much like the earlier example, but now we have a dodecahedron with colored faces instead of a square with colored sides.

To make things a bit easier we will only look at rotations, so patterns that are mirror images of each other will still be considered different. There are four types of rotational symmetries:

A. Rotations around a face by a multiple of 1/5 of a turn.
B. Rotations around a corner by a multiple of 1/3 of a turn.
C. Rotations around an edge by a 1/2 of a turn.
D. The identity.

There are 6 axes through the faces, so 6 axes about which rotation A can take place. Note that a 2/5 turn will keep the same patterns unchanged as a 1/5 turn, because doing it three times give a 3*2/5 = 6/5 turn which has the same effect. Therefore all 6*4 = 24 type A rotations keep the same positions fixed. Under such a rotation, the faces move in two 5-cycles, and two 1-cycles. Each cycle must be all the same color, so there are $2^4$=16 fixed points for these rotations. Similarly the 20 B rotations have $2^4$ fixed points, the 15 C rotations have $2^6$, and finally the identity has $2^{12}$ fixed points.

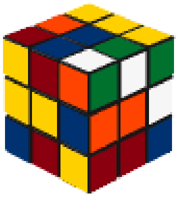
The average is therefore $(24{\cdot}2^4 + 20{\cdot}2^4 + 15{\cdot}2^6 + 2^{12}) / (24+20+15+1) = 96$. There are 96 distinct Orbix positions (if mirror images are counted as well).

If you do the same thing for all mirror symmetries (it is easiest to use a point reflection through the centre of the dodecahedron followed by any normal rotation) then you get the result:

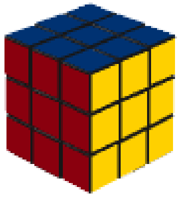
$$(24 \cdot 2^4 + 20 \cdot 2^4 + 15 \cdot 2^6 + 2^{12} + 24 \cdot 2^2 + 20 \cdot 2^2 + 15 \cdot 2^8 + 2^6) / 120 = 82.$$

This means there are 82 positions if mirror images are considered the same, and therefore there are 96-82 = 14 pairs of positions that are mirror images of each other.

Burnside's Lemma can also be applied to the Rubik's cube. Whereas normally there are 12 positions that are a quarter turn from being solved, these are essentially the same. By applying the lemma, we find that there are only 901,083,404,981,813,616 essentially different positions.



# Beginner’s Solution to the Rubik's Cube



## *Introduction*

There are many different methods for solving the Rubik's cube. They can be divided into two broad categories: layer methods and corners first methods (and there are sub-categories within these broad categories). The method I use for speedsolving is a layer based method. More specifically, the method I currently use is: cross, F2L, 3-look LL (I know some of the OLLs, so sometimes I can do a 2-look LL). If you are a newbie cuber then this description may not mean much to you, so I should add that it's the 'Advanced Solution' I described in the **Next Steps** section at the end of this page.

Many years ago when I wrote this webpage there were many great websites that explained advanced and expert methods for solving the cube (check out my **Rubiks links** page), however, there were very few that explained beginner methods. This is the reason I wrote this page. It's not meant to be a totally comprehensive explanation, it's really just some notes I threw together for some friends I was teaching. I thought it might be useful for others, so I've turned it into a webpage.

This beginner method requires memorizing only a few algorithms, and when done efficiently can achieve solves of 60 seconds or faster. I know people who can solve in 20-30s with a method like this. I haven't been able to solve so fast with a beginner’s method, so don't be too distressed if you can't either. On the other hand, if you can do 30s solves with this method, then you are too good for this method and you should be learning an Advanced or Expert method!

Aside from minimal memorization, another benefit of this method is that it is very scalable. More algorithms may be added later to develop it into an advanced method, or if you're really keen, an expert method. This means you don't need to scrap it and start again to move to an expert method. Everything you learn here will be useful for more advanced methods.

## *Structure of the cube*

We all know that 3x3x3 = 27, however, rather than thinking about the cube as 27 little "cubies", think about it as 6 fixed centers (that can rotate on their own axis) with 8 corners and 12 edges which rotate around it. As the centers are fixed, the centre color defines the color for the face. It's important to remember this otherwise you'll end up trying to do illogical (mechanically impossible!) things like wondering why you can't work out how to put a corner piece in an edge position, or assuming that you're looking at the blue face merely because 8 of the 9 cubies on it are blue (if the center is white then it's the white face).

## *Terminology*

When describing the solution for the 2nd and 3rd layers, standard cube notation will be used. Here's what you need to know to read it:

**F** = Front face **B** = Back face **R** = Right face **L** = Left face **U** = Up face **D** = Down face

In addition to a letter, each move may be accompanied by an apostrophe or the number two:

- A letter by itself means turn that face 90 degrees clockwise (e.g. **F**).
- A letter followed by an apostrophe means turn that face 90 degrees anti-clockwise (e.g. **F'**).
- A letter followed by the number 2 means turn that face 180 degrees (direction is irrelevant), (e.g. **F2**).

So **R U' L2** is a shorthand for *"turn the Right face 90 degrees clockwise, then turn the Up face 90 degrees anti-clockwise, then turn the Left face 180 degrees"*. When thinking whether to turn clockwise/anti-clockwise, imagine that you are looking directly at the particular face you are turning.

For each algorithm, the notation is written with the assumption that the core of the cube remains fixed throughout the whole algorithm, and the faces just turn around it. This means that you also need to know how to position the cube to start the algorithm.

For pictures and further detail about cube notation, have a look at **Jon Morris' cube notation page**.

## The Solution

### *The First Layer*

The first layer is solved in two stages:

1. Form the cross
2. Insert the 4 first layer corners (each corner is inserted individually)

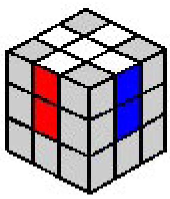
I believe that the first layer should be done intuitively. You need to understand it and solve it without learning algorithms. Until you can do this, I wouldn't bother attempting the rest of the cube! So, spend some time playing with the cube and familiarizing yourself with how to move the pieces around the cube.

Now, here are some tips to get you started.

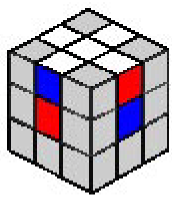
### *The Cross*

I prefer to start with the white cross because I find white easier to quickly identify on a completely scrambled cube, however, you can use any color.

There are 4 edge pieces with white (i.e. the 4 arms of the cross) which have specific positions. You can't put any white edge piece in an arm of the cross because the other color on the edge cubie must match up with it's center on the middle layer.



Here is a pic of what a correctly formed cross looks like (grey denotes cubies that are irrelevant to the cross). Note that the **white**/**red** edge cubie matches up with the **white** centre and the **red** centre. Ditto re the **white**/**blue** cubie.

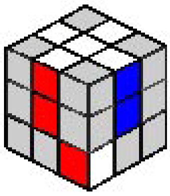


Here's a pic on an incorrectly formed cross. Looking at the **white** face we do indeed see a **white** cross, however the **white**/**red** edge cubie does not match up with the **red** centre. Ditto re the **white**/**blue** cubie. This is bad!

For a detailed explanation of the cross, check out Dan Harris' **Solving the Cross** page.

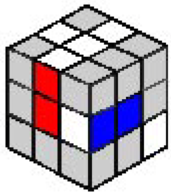
### *The First Layer Corners*

Once you have completed the cross, completing the first layer requires inserting each of the 4 corners in separately. The first thing to do is examine your cube and locate all of the top layer edge pieces - they will be sitting in either the first layer or the last layer. Inserting the first layer corners should be done intuitively, not by learning algorithms. To get you started, here's a step-by-step example of one way to insert a first layer corner.



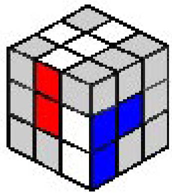
**Step 1**

The **blue**/**red**/**white** corner is sitting in the bottom layer (the **blue** part is facing the bottom so we can't see it in this picture). Turn the **blue** face 90 degrees anti-clockwise.



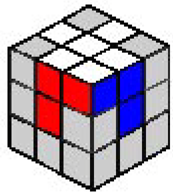
**Step 2**

Now your cube should look like this. Move the D face 90 degrees anti-clockwise to line up the **blue**/**white** edge with the **blue**/**white**/**red** corner.



**Step 3**

Now that the **blue**/**white** edge and the **blue**/**white**/**red** corner have been lined up, re-form the **white** cross by turning the **blue** face 90 degrees clockwise.

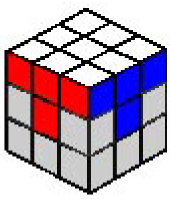


**Step 4**

Now the **blue**/**white**/**red** corner is in its correct place.

*Here are some tips for inserting the top layer corners:*

- Start with a first layer corner that is sitting in the last layer.
- If there are multiple first layer corners in the last layer (there usually will be), start with one that does not have the white part of the corner on the face opposite the white face. Or, if you were using a different color for the cross ('color X'), start with a corner that does not have the 'color X' part of the corner on the face opposite the 'color X' face.
- When working with a first layer corner piece that is in the first layer (but in the wrong first layer corner position), you will need to get it out of the first layer into the last layer, then insert it into the correct first layer corner position. The same principle applies if a first layer corner piece is in the correct first layer corner position but needs to be flipped around. You need to get it out of the first layer (i.e. into the last layer), and then re-insert it into the first layer the correct way around.



This is what the first layer should look like when finished.

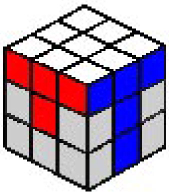
### *The Middle Layer*

The middle layer consists of one stage:

1. Insert the 4 middle layer edges (each edge is inserted individually).

You only need to learn one algorithm (plus the mirror algorithm) for the second layer. There are many more algorithms, but let's just learn the essential one first.

First, locate a middle layer edge that is currently sitting in the last layer. I'm going to use the **blue**/**red** edge for this example.



This blue edge cubie in the last layer is the **blue**/**red** edge cubie.

In this picture, **U=white, L=red and F=blue**. We can't see the other three faces, but obviously the R face is the one opposite the L face, the D face is opposite the U face and the B face is opposite the F face.

Now, position the **blue**/**red** edge piece so that the color on the side of the cube (**blue** in this case) lines up with it's center. Now perform the following algorithm: **D L D' L' D' F' D F**

If the **blue**/**red** edge piece was flipped the other way so that the **blue** was on the bottom rather than the **red**, you would position the cubie under the **red** centre and perform the following algorithm: **D' F' D F D L D' L'**. This is the mirror of the previous algorithm. The axis of symmetry lies diagonally across the white face, and along the line which divides the **blue** face and the **red** face.

***What if the edge piece is not in the last layer?***

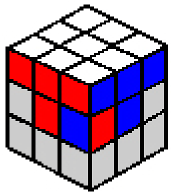
The instructions above assume that the middle layer edge piece you are inserting is sitting somewhere in the last layer.

If some middle edges are in the last layer and some are in the middle layer in the wrong spot, always start working with the edge pieces that are in the last layer. After you've done this, sometimes (but not too often) you'll be left with a middle layer edge piece that's in the middle layer but in the wrong spot. In this situation, you can use the same middle layer algorithms from above:

(**D L D' L' D' F' D F** or **D' F' D F D L D' L'**)

to insert another edge piece into the middle layer edge position, thereby knocking the middle layer edge piece out of its spot and into the last layer. Once you've done this, the middle layer edge piece is in the last layer and you can deal with it in the usual way.

There is a short-cut to this problem, but as this is a beginner solution with minimal memorization, I haven't included it here. If you really want to learn it, take a look at Case Dd2 on Dan Harris' site.



*The **red**/**blue** middle layer edge piece is in the middle layer but not oriented correctly. It needs to be moved to the last layer, then put back into the middle layer in the right way.*

## The Last Layer

The last layer ("LL") is done in 4 steps:

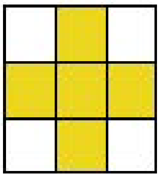
1. Orient the edges (2 algorithms) - i.e. form a cross on the D face.
2. Permute the corners (1 algorithm) - i.e. get the corners in the correct position in 3D space (don't worry if they still need to be rotated).
3. Orient the corners (1 algorithm + 1 mirror algorithm) - i.e. flip the corners.
4. Permute the edges (1 algorithm) - i.e. swap the edges around. The cube should now be solved! :)

All last layer algorithms are performed with the cross (i.e. the first layer - white side in this example) on the bottom.

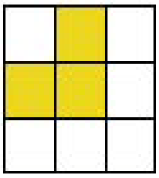
## *Orienting the LL Edges*

Once you have completed the first two layers ("F2L"), hold the cube so that the white side is on the bottom. The white side will be on the bottom for the remainder of the solution. This means that the white side is the D side for all last layer algorithms.

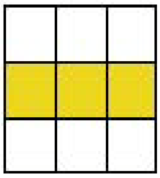
On my cube, white is opposite yellow, therefore yellow is the U face for all last layer algorithms on my cube. Note that your cube may have a different color opposite white (e.g. blue). Now have a look at your last layer, and in particular, look at the last layer face - there are 4 possible patterns of LL edges that you may see.



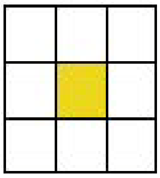
**State 1**



**State 2**

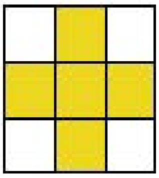


**State 3**



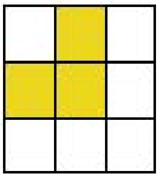
**State 4**

Unlike with the initial cross (where all the edges must match up with the white center and with the centers on the middle layer), here all you need to worry about is getting all the last layer edges matching up with the last layer center. It doesn't matter if the other color on the LL edge piece does not match up with the color on the middle layer center. Also, ignore the LL corners too. It doesn't matter what they are doing at the moment. Now, let's consider each of these LL edge states separately.



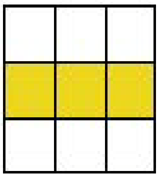
**State 1**

All the edges are already oriented correctly. Move on to permuting the corners.



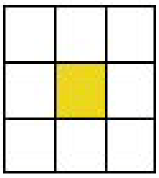
**State 2**

We are going to re-orient our faces for this algorithm. The face you are looking directly at in this picture is now the U face (it was the D face for when you were doing the second layer edges). Perform the following algorithm: **F U R U' R' F'**



**State 3**

As with State 2, the face you are looking directly at in this picture is now the U face. Perform the following algorithm: **F R U R' U' F'**



**State 4**

State 4 is really a combination of States 2 and 3, so all you need to do is perform the algorithm for either State 2 or State 3. Once you've done this, you'll see that your LL edges now look like State 2 or State 3, so just perform the appropriate algorithm and you will have a cross on the LL.

## *Permuting the LL Corners*

The two possible states are:

- two adjacent LL corners need to be swapped; or
- two diagonal LL corners need to be swapped.

These are the only two possible states. If you cannot identify one of these two states with your LL corners then one or more of the following must be true:

- You have not finished the F2L.
- Someone has ripped out a corner of your cube and put it in the wrong way.
- Someone has ripped off some of your stickers and put them back in the wrong place.
- You are not looking hard enough. ;)

### *Swapping adjacent corners*

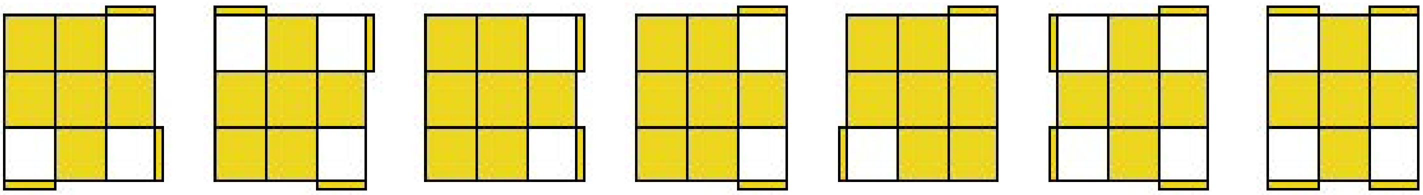
Hold the cube with the white side on the bottom, and the two corners to be swapped are in the front right top and the back right top positions. Perform the following algorithm: **L U' R' U L' U' R U2**. To see an animated version of this algorithm, see the first algorithm on [Lars Petrus' Step 5 page](). On Lars' site, the algorithm is being executed from a slightly different angle (the two corners being swapped are front-top-right and front-top-left), but it is the same exact algorithm.

### *Swapping diagonal corners*

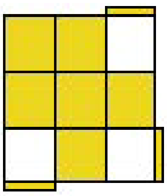
Swapping diagonal corners can be done by executing the adjacent corner swap algorithm twice. Perform it once to swap any two LL corners. Re-examine you cube and you'll see that now there are just two LL corners that need to be swapped. Position it correctly for the final LL adjacent corner swap and perform the LL adjacent corner swap algorithm.

## *Orienting the LL Corners*

There are 8 possible orientation states for the LL corners. One is where all 4 corners are correctly oriented. The other 7 look like this.

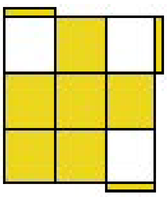


**State 1** **State 2** **State 3** **State 4** **State 5** **State 6** **State 7**



### *State 1. Twisting three corners anti-clockwise*

**R' U' R U' R' U2 R U2**



### *State 2. Twisting three corners clockwise*

**R U R' U R U2 R' U2**
To see an animated version of this algorithm, look at [Lars Petrus' Sune algorithm]().

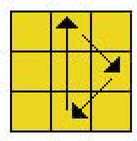
### *States 3-7*

Once you know the algorithms for States 1 and 2, you can solve any LL orientation State. The remaining States can be oriented using a maximum of 2 algorithms. You will need to do one of the following (i) the State 1 algorithm twice, (ii) the State 2 algorithm twice, (iii) the State 1 algorithm, then the State 2 algorithm, or (iv) the State 2 algorithm, then the State 1 algorithm.

In a previous edition of this solution, I had said that I'm not going to tell you exactly how to combine the State 1 and State 2 algorithms to solve States 3-7. My reason for this was because it is important that you try to understand how the State 1 and the State 2 algorithms work, and that once you do understand them you will be able to work out how to use them to solve all the States. I still believe this, however, I received emails from a few people who were having trouble with States 3-7, so I decided to write some extra tips. I still suggest that you try to work out States 3-7 by yourself, but if you are really stuck, have a look here: Orienting the Last Layer Corners: further tips.

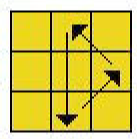
### *Permuting the LL Edges*

There are 5 possible permutation states for the LL edges. One is where all 4 edges are correctly permuted. The other 4 look like this.



**State 1**

**R2 U F B' R2 F' B U R2**

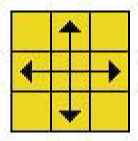


**State 2**

**R2 U' F B' R2 F' B U' R2**

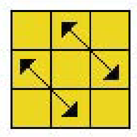
This is almost identical to the algorithm for State 1. Only difference is the 2nd move and the 2nd last move.

For an animated version of this algorithm, see the Lars Petrus' Allen algorithm. The algorithm is being executed from a slightly different angle, but it is the same exact algorithm.



**State 3**

Apply the algorithm for either State 1 or State 2. Re-examine your cube and it will now look like State 1 or State 2.



**State 4**

Apply the algorithm for either State 1 or State 2. Re-examine your cube and it will now look like State 1 or State 2.

**And that's all you really need to know to solve the Rubik's Cube!** With practice, you should be able to achieve times of 60 seconds (or faster) using this method. Once your comfortable with this method and want to learn more, take a look at the following section.

## *Next Steps*

If this beginner method is too easy and boring for you then consider the following.

### ***Intermediate method***

- Solve each first layer corner + corresponding middle layer edge in one step. This means that after the cross you only have 4 steps (4 corner/edge pairs) to complete the F2L. With this beginner method there are 8 steps: solve each of the 4 first layer corners, then solve each of the 4 middle layer edges. I'd suggest just playing around with your cube and figuring out the F2L corner/edge pairs yourself. For some hints about solving the F2L intuitively, have a look at **Doug Reed's intuitive F2L guide**. If you're still stuck and just want the algorithms, check out **Dan Harris' F2L page** and **Jessica Fridrich's F2L page**.

- Learn the 4 specific algorithms (or rather, 3 algorithms plus one mirror algorithm) for each of the 4 different permutation states of the LL edges. My beginner solution already shows you 2 of the 4 last layer edge permutation algorithms, the other two last layer edge permutation algorithms are Case #5 and Case #17 on **Dan Harris' PLL page**.

### ***Advanced method***

- Learn everything from the Intermediate method.
- Learn the 3-look LL. This requires learning the 7 specific algorithms for the 7 different orientation states of the LL corners, and learning the 21 PLL algorithms (permuting the last layer algorithms) so you can permute the LL edges and LL corners at the same time. A full 3-look LL uses 30 algorithms.

For more details about the advanced method, check out t **Rubiks Galaxia 3-look LL**, **Dan Harris' site** and **Lars Vandenbergh's PLL page**.

### ***Expert method***

- Do the F2L in 5 steps (first dot point from the Intermediate method).
- Learn a full 2-look LL. This requires memorizing 21 PLL algorithms, plus 57 OLL algorithms (orienting the last layer algorithms).

For more details about the expert method, check out **Dan Harris' site**, **Joël van Noort's site** and **Lars Vandenbergh's site**.

## *Other stuff*

The method I've documented here is what I believe to be a good beginner's method. The problem with some beginner's methods is that they are not scalable - to improve your cubing you have to un-learn much of what you know and re-learn it in a different way. This method focuses on memorizing very few algorithms, but is structured in a way that allows for development into an intermediate or advanced method. Other thing I should say is that I didn't actually devise any of the last layer algorithms in this method. I merely chose a selection of existing algorithms (sourced from a variety of places including **Jessica's site** and **Dan K's site**) and organized them into a simple solution method.

## *Celebrate your cubing success!*

When you are confident that you can solve the cube by yourself, time yourself so you can keep track of your progress. Also, consider submitting your time to the **unofficial world records**.

# THE ULTIMATE SOLUTION TO THE RUBIK'S CUBE

## THE EDGE PIECE SERIES

**In this approach to solving a scrambled Rubik's Cube all 12 edge pieces are placed first. The first four are placed in Step One which is straightforward. Some of the others are moved into place with a simple replacement process. The remainder utilize a symmetric four turn series which moves three edge pieces around a corner of the cube.**

**The red face of the cube in Fig. 1a is front.The red/yellow edge piece at front/top belongs at front/right. We cannot simply rotate the front face clockwise by 90$^o$ because we have already placed four blue edge pieces on the bottom face and would not want to move any one of them out of place. But an Edge Piece Series will move the red/yellow, orange/yellow and red/green edge pieces around the front/right/top corner of the cube without moving other edge pieces. (These edge pieces have been numbered #1, #2 and #3.) According to standard notation this series is F R' F' R.**

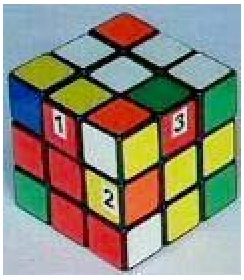



**Fig. 1a**

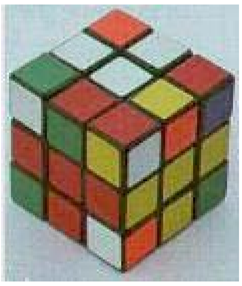


**Fig. 1b**

**Applying this series gives the cube shown in Fig. 1b. We find that edge piece #1 is now in the position originally occupied by edge piece #2. That is, #1 replaced #2. Further, #2 replaced #3 and #3 replaced #1. The three pieces moved about the front/top/right corner of the cube in a counterclockwise direction. The blue edge pieces on the bottom of the cube are undisturbed. Some corner pieces have also moved but that is immaterial. At this time we are interested only in the movement of edge pieces.**

**We can also describe this series in terms of the movement of edge pieces #1, #2 and #3.**

| | | |
|---|---|---|
| **Turn one** | **Front clockwise** | **#1 replaces #2** |
| **Turn two** | **Right counterclockwise** | **#3 replaces #1** |
| **Turn three** | **Front counterclockwise** | **reverse turn one** |
| **Turn four** | **Right clockwise** | **reverse turn two** |

**Now let us consider what has happened to these edge pieces. If the front face in Fig. 1b is turned 90$^o$ counterclockwise we find that edge piece #1 (red/yellow) has the same orientation as before. The same is true of edge piece #2 (turn the right face 90$^o$ counterclockwise). But edge piece #3 is different. If we turn the top face 90$^o$ counterclockwise we find that edge piece #3 has been inverted.**

**This is always true of these edge pieces. Edge pieces #1 and #2 are not inverted while edge piece #3 is inverted. You are free to number the three edge pieces in any way you want to. Then, provided you always follow the turn sequence given above, you will find that #1 replaces #2, #2 replaces #3 and #3 replaces #1. Edge pieces #1 and #2 will not invert while #3 inverts.**

**Look at the cube in Fig. 2a. This is the same cube as in Fig 1a except that the top face has been turned 90$^o$ in a counterclockwise direction. We still want the red/yellow edge piece to move into**

the front/right position. But to be properly placed it must invert. Hence it must be edge piece #3. Since #3 replaces #1 then orange/yellow must be edge piece #1. Green/white then is edge piece #2.

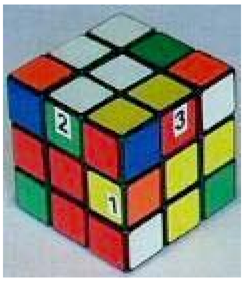



Fig. 2a

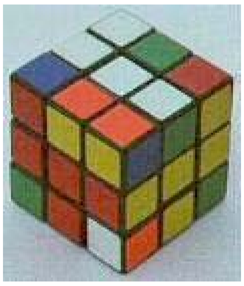


Fig. 2b

We will apply the series as before.

| | |
|---|---|
| turn one | #1 replaces #2 |
| turn two | #3 replaces #1 |
| turn three | reverse turn one |
| turn four | reverse turn two |

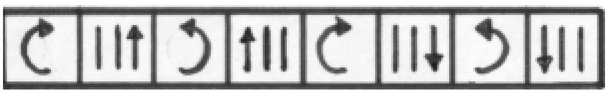
Now examine the result, the cube in Fig. 2b. Again we have accomplished our main purpose, the movement of the red/yellow edge piece into its proper position and orientation at front/right. The orange/yellow piece is involved in both instances but it moves to a different location. However, red/green is involved in the first case while the green/white piece has replaced it in the second series. In the second example, the three pieces have moved in a clockwise direction about the front/right/top corner of the cube.

The red/yellow edge piece may be moved into its correct position/orientation from a position above either the red or the yellow center pieces by an Edge Piece Series. Which version of the series you will want to use will depend on what else you are trying to accomplish. We will visit this situation again in Step Two of the Ultimate Solution to Rubik's Cube.

## THE CORNER PIECE SERIES

The corner pieces are put into proper position/orientation following the placement of the 12 edge pieces. After the edge pieces are placed we will find some corner pieces may already be correctly placed while some are properly positioned but not properly oriented. The remainder will be in the wrong position. The incorrectly placed corner pieces will be moved into proper position/orientation using the Corner Piece Series.

The Corner Piece Series has eight 90$^o$ turns and is perfectly symmetrical. The first turn is always a turn of the top face. Such a turn is indicated by a curved arrow on the top face of the cube with the point of the arrow ending on the back part of this face. The first figure in the set of eight below indicates that the top face is rotated clockwise. We will say that this face is turned to the right because the arrow at the back of the top face points to the right. This is simply a convention but it will enable you to more easily recall this series. The symbols below indicate the eight turns in one example of the series. You are looking down on the top face of the cube.



The first move of the top face may be either to the right or to the left , but the rules governing the succeeding moves are the same in either case. The second symbol shows that the top surface of the right face moves away from you.

1. The first move is a turn of the top face, either to the right or to the left.

2. The second move is a turn of the side face toward which the top was turned. It is turned away from you.

3. The top is then turned in the direction opposite to the first turn.

4. The side toward which the top was just turned is then turned away from you.

5. The top is then turned in the same direction as the first turn.

6. The side toward which the top was just turned is then turned back toward you.

7. The top is turned in the opposite direction.

8. The side toward which the top was just turned is turned back toward you.

The top is turned every other turn, alternating directions. Each side face is first turned away and then back toward you. Note that each of the initial turns is later reversed.

This series will cause three corner pieces on the top face of the cube to move about a triangle. If the first turn of the top face is to the right then the corner pieces will move in a counterclockwise direction about the triangle. If the first turn is to the left then three pieces (two are the same but one is different) will move in a clockwise direction about a triangle which is a mirror image of the first triangle.

Let us apply a Corner Piece Series (with the first move being a turn of the top face to the right) to the cube in Fig. 3a. In this case the orange face is the front of the cube. The result is the cube in Fig. 3b. We can see that the left/front/top, the left/back/top and the right/back/top pieces have been moved. We get a another view of the changes by rotating the cube 180$^{o}$ giving us Fig. 3c.

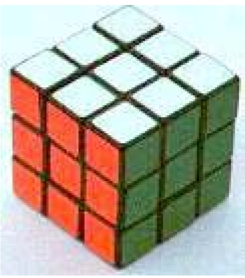


Fig. 3a

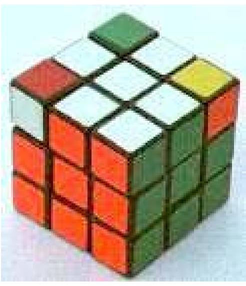


Fig. 3b

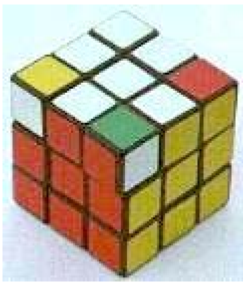


Fig. 3c

The red/green/white corner piece moved along the red/white edge and into the red/yellow/white corner. In the process it "rolled over" so that the white face is now on the side of the cube and in the yellow face. The red/yellow/white corner piece moved along the yellow/white edge and also "rolled over". The yellow/orange/white piece moved across the diagonal of the top face. In the process it's yellow face, which had been on the left (yellow) side, came to the top.

Note that the two corner pieces on the back of the top face and the corner piece at left/top/front are involved in this triangle. The pieces moved counterclockwise. All three "rolled over".

If the first turn of the top face had been to the left the two pieces at the rear would still be involved but the third piece would be the one in the front/right/top corner. This piece would

**move across the diagonal to the left/back/top corner. The pieces would move in a clockwise direction about the triangle.**

**Obviously the two pieces at the back of the top face are always involved along with one of the pieces at the front. How can you tell which one? You should note that the first turn immediately moves the front corner piece involved to the back of the top face. It is this piece which will eventually move across the diagonal of the top face.**

**If the three out-of-position corner pieces were to move in the opposite direction about the same triangle they would move back into position/orientation. These three pieces illustrate what you will be looking for as you try to solve a scrambled cube.**

**The red/white/green piece may be made to move along the red/white edge of the cube. As it does so it will roll over into its proper position/orientation. The yellow/white/red corner piece will do the same thing as it moves along the yellow/white edge of the cube.**

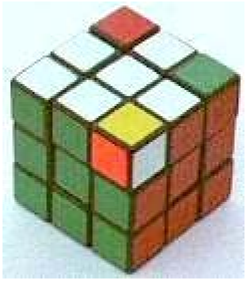


**Fig. 3d**

**The cube has been turned another 90$^{o}$ in Fig. 3d. Now you see an example of a corner piece which can be moved across the face diagonal to its proper position and orientation. For this corner piece, the color of the top face (white) is on the right side. The orange and yellow colors do not match any side which is visible. But you should know that orange is opposite red and that yellow is opposite green. Hence this corner piece has the colors of the opposite corner.**

**The Corner Piece Series which begins with a turn of the top face to the left will move the orange/white/yellow corner piece across the diagonal with the white face coming to the top. The other two pieces move along cube edges and roll over into their proper position and orientation. The cube is again complete.**

**Of course you will not often find three corner pieces arranged so that all move into correct position/orientation in a single series. Most of the time you will find one which can move into place along a cube edge. Less often it will be a corner piece which can move across a face diagonal and into its proper position and orientation. At times there will be two pieces which can be moved into place in a single series and rarely three. But if you should find three in such an arrangement you certainly would want to take advantage of that fact.**

## STEP ONE: THE CROSS

**This solution to Rubik's Cube begins, as many solutions do, with the formation of a cross on one face of the cube. You can choose any color but it is best to choose a face which already has an edge piece matching the center color of that face. The scrambled cube shown in Fig. 4a has three faces for which this is true. I have chosen the blue face but any one of the three could have been chosen. (On my cube, red is opposite orange, yellow is opposite green and white is opposite blue.)**

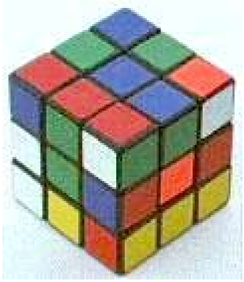


Fig. 4a

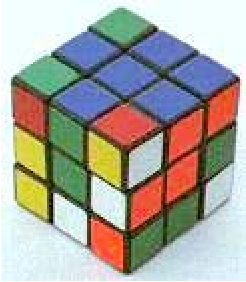


Fig. 4b

**Naturally it would be better if two edge pieces had their blue color on top but only if these two pieces are in the proper position relative to each other. More often than not this is not true and one edge piece would have to be changed. In two trials out of three, two will not be better than one.**

**What is the best way to transform the cube to the one in Fig. 4b? Many solutions would tell you to begin by turning the top (blue) face one turn (90°) clockwise. The blue/green piece would move into its proper place on the cube. These solutions emphasize the bottom face. Target edge pieces are taken to the bottom of the cube, the bottom is turned and the target piece brought to the top in proper position and orientation. That approach would require eight moves to form the cross on the blue face of this cube.**

**But it is better to emphasize the top face. Edge pieces are moved to the top face so as to be in the proper relationship to top edge piece(s) already on the top face. First, examine the cube. The blue/green edge piece is already on the top face; the blue/red piece is at front/right; the blue/yellow piece is at bottom/front; and the blue/orange piece is at back/bottom.**

**The blue/yellow piece must be opposite the blue/green piece. In addition, the blue/red piece belongs where the green/orange piece is now. If the blue face is rotated 90° <u>counterclockwise</u> then a 90° clockwise rotation of the orange face will move the blue/red edge piece into place relative to the blue/green piece and a 180° rotation of the green face moves the blue/yellow piece into place, also relative to the blue/green piece.**

**Three edge pieces are in place on the blue face. We can now rotate the blue face by 180° and then the yellow face by 90° clockwise. Unfortunately this moves the blue/yellow edge piece out of position. After the blue/orange piece is moved into position/orientation by a 90° counterclockwise rotation of the orange face, the blue/yellow edge piece is moved back into place by a 90° counterclockwise rotation of the yellow face. This approach required seven moves.**

**But it is still better to <u>anticipate</u>. The first move should have been a 90° counterclockwise rotation of the yellow face. This moves the blue/orange edge piece into position so that, following placement of the blue/red and blue/yellow edge pieces, a clockwise rotation of the red face moves the blue/orange piece into its proper position/orientation relative to the other three. Finally, a 180° rotation of the blue face moves all blue edge pieces into place. This approach requires six moves. In standard notation the moves are (back is B):**

**B' T' R $F^2$ L $T^2$**

**In actual practice, of course, how you proceed to form the cross is up to you. Just make sure that you can do it somehow. With sufficient practice you should be able to reduce the average number of moves required to about seven.**

## STEP TWO: CENTER SECTION EDGE PIECES

**In Step Two of The Ultimate Solution to Rubik's Cube you will have two objectives which are to be met simultaneously. You will use the Edge Piece Series to move a central section edge piece**

from the top face into its proper position and orientation in the center section. You will repeat this process until three of the center section edge pieces are in place.

As you carry out this process some of the pieces you will be moving will be top section edge pieces. Make sure you carry out the Edge Piece Series so that, if possible, the top edge pieces wind up with their top color on the top face of the cube. This step is complete when:

a. three edge pieces are in place in the center section,

b. three top edge pieces have their top color on the top face of the cube,

c. one top edge piece is in the center section and

d. the fourth center section edge piece is in the top section of the cube.

In the beginning it is possible that one or more center section edge pieces are already in place. Fine. But it is also possible that one or more are in the proper position but with the wrong orientation. Tough. You will have to remove it (them), also with an Edge Piece Series.

Remember that a center section edge piece may be moved into proper position and orientation with that edge piece starting above either of the faces with one of the edge piece colors. The orange/green edge piece in Fig. 5a (at top/right) could be moved into place at front/right if it is treated as Edge Piece #1. But this would be a bad idea because the white/red edge piece would be inverted in the process and we would have reduced the number of top edge pieces with their top color on top of the cube from one to none.

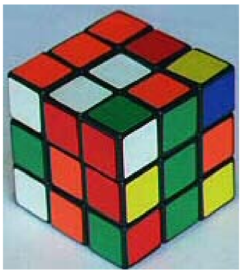


Fig. 5a

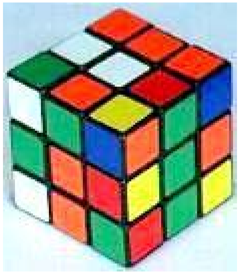


Fig. 5b

On the other hand we could treat the orange/green edge piece as Edge Piece #2 in Fig. 5a (red/white would be #1) or rotate the top face 90$^{o}$ clockwise (giving Fig. 5b) and treat orange/green as Edge Piece #3. The red/yellow edge piece would be #1. In either case the orange/green edge piece moves into the proper position and orientation and the red/white edge piece will continue to have its white color on the top face of the cube.

But we need to get more white edge piece faces on the top face of the cube. In Fig. 5c the red/green edge piece belongs at front/right. If we treat the red/green edge piece as #1 an Edge Piece Series will cause the orange/white edge piece to come to the top of the cube with its orange face on top, not white. On the other hand if we treat the yellow/red edge piece as #1 then red/green is #2 and orange/white would be #3. Orange/white would come to the top of the cube with its white face on top (since edge piece #3 always inverts).

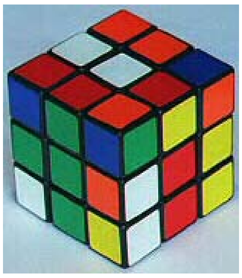


Fig. 5c

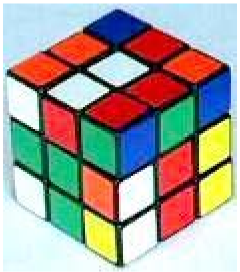


Fig. 5d

**Or we could rotate the top face 90° counterclockwise (giving Fig. 5d) and apply an Edge Piece Series with the red/green edge piece as #3. The orange/white edge piece will be #1 and the red/white edge piece will be #2. The orange/white edge piece will come to the top of the cube with its white face on top and the white face of the red/white edge piece will remain on top. We will now have two white faces on the top face of the cube.**

**Continue this process until three center section edge pieces are in place, three top edge pieces have their top color on top and the fourth top edge piece is in the center section. The fourth center section edge piece will be on the top of the cube.**

**If you should, by chance, properly place three center section edge pieces but have only two top edge pieces with their top colors on top, then apply one more Edge Piece Series in which all of the pieces involved are top edge pieces. The top edge piece in the center section comes to the top with its top color on top and the top edge piece which does not have its top color on top moves into the center section.**

**You should note that you do not always have to complete the fourth turn of the Edge Piece Series. Whenever the fourth turn is a turn of the top face it does not need to be made because a turn of the top face can have no effect on the bottom face. Remember, the Edge Piece Series was devised not only to put center section edge pieces into proper position/orientation but also to prevent the movement of edge pieces on the bottom face.**

## STEP THREE: THE TOP EDGE PIECES

**In this section we will place the fourth center section edge piece and the four top edge pieces in their proper positions and orientation.**

**At the conclusion of Step Two, three top edge pieces were to have their top color on the top face of the cube. Those three edge pieces could be in any one of the following possible arrangements:**

**1. All three could be out of order.**

**2. Two adjacent pieces could be in the proper order relative to each other with the third out of order.**

**3. Two opposite pieces could be in the proper order relative to each other with the third out of order.**

**4. All three could be in the correct order relative to each other.**

**The three white edge pieces on the top of the cube in Fig. 6a are all out of order. Note that the top edge piece face you cannot see on the back of the cube must be red. If the top face of the cube is turned 90° clockwise then the yellow face of white/yellow would be adjacent to the yellow face center and white/yellow would be in the proper position/orientation.**

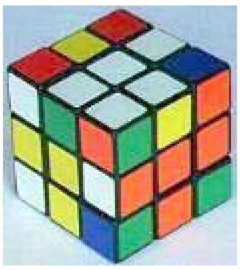


**Fig. 6a**

**But neither of the other two white edge pieces would be in the correct position. The green color would be above the red face center and the red color would be above the orange face center. No matter which of the three top edge pieces is put into place the other two will be out of place.**

**We can correct this by applying an Edge Piece Series to the cube. However, the series must involve three top (white) edge pieces. Rotate the top face until three white edge pieces are adjacent to the top/front/right corner as they are in Fig. 6a. The cube obtained by rotating the top face 90$^o$ clockwise would also be suitable.**

**Always be sure that three white pieces are in the series and that it concludes with three white faces on top of the cube. In all cases the white edge piece in the center section will be edge piece #1. In Fig. 6a white/orange will be #1 and white/yellow will be #2. Any other choice would not leave three white edge pieces with their white faces on top of the cube at the end of the series. This series will change the top edge pieces to one of the other arrangements.**

**In the case of the cube in Fig. 6a the series converts it to the cube in Fig. 6b. Here two adjacent edge pieces are in the proper order (white/yellow and white/orange) and one is out of order (white/red where red is above green). These pieces may be put in the proper order by turning the top face by 180$^o$ and applying an Edge Piece Series to the top edge piece in the center section (white/green), the top edge piece which is out of order on the top face (the white/red piece) and the fourth center section edge piece (yellow/orange). The latter is moved into proper position/orientation and all four top edge pieces will be in their proper order. A 90$^o$ counterclockwise turn of the top face will put all top edge pieces in their proper place. Step Three is complete.**

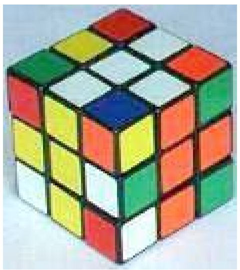


**Fig. 6b**

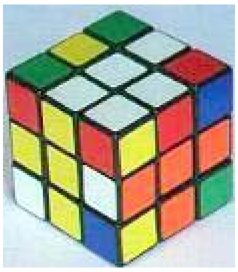


**Fig. 6c**

**In the third case two opposite edge pieces are in position relative to each other and the third is out of place as shown in Fig. 6c. The white/yellow top edge piece is in place. By the process of elimination the color you cannot see on the white edge piece is green. Since the green face on my cube is opposite the yellow face the white/green piece must also be in its proper place. But the white/red piece is out of place. It belongs on the other side of the top face.**

**The white/orange piece belongs where the white/red is now. The former can be put there by a simple 90$^o$ rotation (clockwise) of the orange face. Turn the top face by 180$^o$ and a 90$^o$ counterclockwise turn of the orange face will move the white/red edge piece back to the top face and into its proper position relative to the other three. Finally another 180$^o$ turn of the top face moves all white edge pieces into place. Step Three is complete.**

**In the fourth case all three top edge pieces on the top face are in their proper position relative to each other. This is shown in Fig. 6d. The unseen color of the white edge piece is red and it clearly is in the red face of the cube. This arrangement is converted to one in which all four white edge pieces are in the proper order on the top face by a series of replacements.**

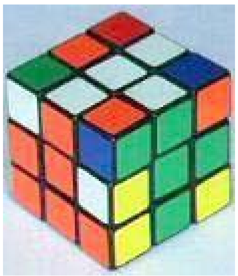
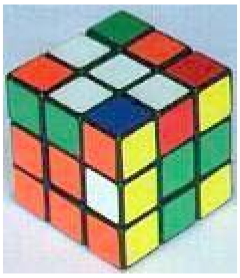
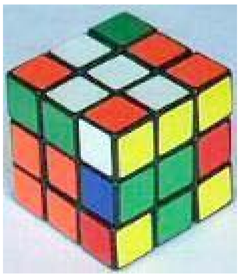
  

**Fig. 6d** **Fig. 6e** **Fig. 6f**

**We begin by replacing a white edge piece on either end of the three on the top face with the white edge piece in the center section. For example, rotate the top face in Fig. 6d by 90$^o$ clockwise giving Fig. 6e. Rotate the green face 90$^o$ clockwise giving Fig. 6f. The white/red piece is replaced by the white/yellow piece and the former moves to the center section.**

**Now turn the top face 90$^o$ counterclockwise (Fig. 6g) and replace the white/green piece with the white/red piece (i.e. rotate the green face by 90$^o$ counterclockwise) to give the cube shown in Fig. 6h. Continue in this fashion until, in the fourth turn of the green face the last white edge piece (in this case the white/orange piece) returns to the top face of the cube and the fourth center section edge piece moves into place at front/right.**

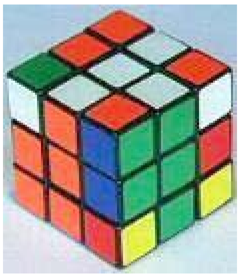
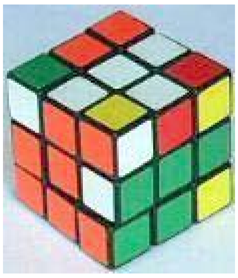
 

**Fig. 6g** **Fig. 6h**

**You might ask how one knows that the fourth center section edge piece has the proper orientation and is not inverted. While you made sure that the top edge pieces had their white color on top you did nothing to ensure that the center section edge piece had the proper orientation. This is true for each of these examples. But you don't need to worry about the 12th edge piece. If 11 edge pieces are in the proper position and orientation then the 12th must be in the proper position and have the proper orientation as well. It cannot be otherwise.**

**Turn the top face so that all top edge pieces are in position (in this case 90$^o$ clockwise). Step Three is complete and all edge pieces have been properly positioned/oriented.**

## STEP FOUR: FIVE CORNER PIECES

**Once the edge pieces have been correctly placed you will look for corner pieces which can be moved into position/orientation by a Corner Piece Series. In the discussion of that series we learned that one or more corner pieces will, almost inevitably, be situated so that this series will move that corner piece along a cube edge and into position/orientation. Others will be in a position, such that the series will move them across a face diagonal and into the correct position and orientation.**

**Figure 7a shows the first kind of corner piece in the front/top/right corner of the cube. The proper series will move the green/red/white corner piece along the green/red edge of the cube and into its correct position and orientation at the back/top/right corner of the cube. (That is, in the green/red/white corner.) In orienting these cubes we will always use the same convention. We will always show the front, right and top of the cube. In this case that means blue is the front, red is the right side and green is the top.**

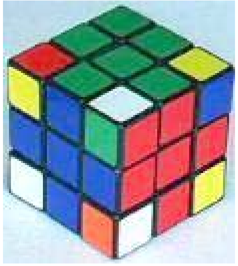
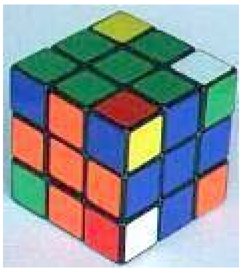
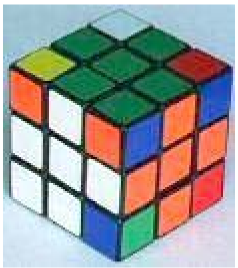
  

**Fig. 7a** **Fig. 7b** **Fig. 7c**

**If the entire cube is turned 90$^o$ counterclockwise we obtain the cube in Fig. 7b. Now orange is front and blue is right. A standard Corner Piece Series which begins with a turn of the top face to the right will move our target corner piece to the back/top/left corner. It will be correctly placed in the green/red/white corner of the cube.**

**Turning the cube another 90$^o$ counterclockwise gives the cube in Fig. 7c. The Corner Piece Series is the same (it begins with a turn of the top face to the right) but the target corner piece now moves along the left/top edge rather than the back/top edge. You can use either orientation of the cube and still move the green/red/white corner piece into place. The difference is that the blue/green/orange corner piece is involved in moving about the first triangle and the blue/red/yellow corner piece has replaced it in the second case.**

**The red/yellow/blue corner piece is at front/top/right in Fig. 7d. This corner piece belongs somewhere on the top face because of its blue color. But the red and yellow colors do not match the other two sides (front and right) which we can see. Yellow is opposite green and red is opposite orange so this red/yellow/blue corner piece belongs in the back/top/left corner of the cube.**

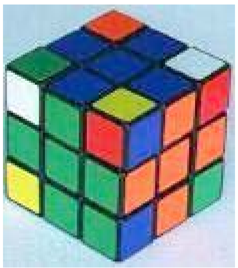


**Fig. 7d**

**A Corner Piece Series which begins with a turn of the top face to the left will cause the red/yellow/blue piece to move across the diagonal of the top face and into place in the back/top/left corner. In this case there is only one way this can happen.**

**If possible, one would like to place two corner pieces at the same time. You may find, by chance, that a piece goes along an edge while another goes across the face diagonal at the same time. Or perhaps two may move along adjacent cube edges. But most of the time you will have to arrange for these things to happen.**

**While other solutions will use a relatively large number of series in solving a scrambled cube this method is able to solve a cube while using only two series because of the use of preliminary face turns which change the corner pieces involved in a Corner Piece Series.**

**Turning the orange face in Fig. 7c by 180$^o$ gives Fig. 7e. We apply the same Corner Piece Series as before (start with a turn of the top face to the right) but the blue/red/green piece has become a part of the triangle. This is the piece which belongs at back/top/left and it is not only moved to the correct position it is also properly oriented. Two corner pieces have moved into place at the same time. Following this series the orange face is again turned 180$^o$ returning it to its original position.**

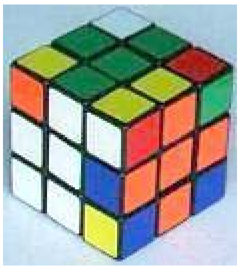
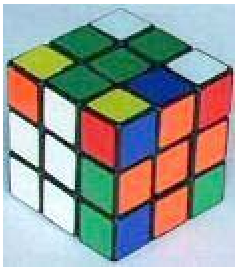


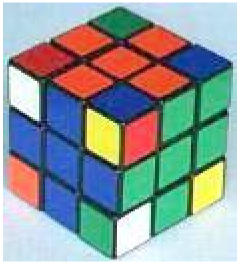
Fig. 7e

**The problem with this approach is that you will have to search for the new piece without any clue as to where it will be found. Rather than trying to find the piece which will replace your target piece, it is better to work with the piece which the target piece is going to replace. This is true because a glance at the latter piece will immediately tell you where it must go.**

**The target piece in Fig. 7c will replace the orange/yellow/blue corner piece at front/top/left. The Corner Piece Series (first turn is to the right) will move this piece across the diagonal with its blue face coming to the top. But where does the orange/yellow/blue corner piece belong? Since none of its colors match any of the faces it touches it must belong in the opposite corner of the cube. And more specifically its blue color must be in the blue face of the cube.**

**The front of the cube is white so the back is blue. Turn the orange face (in Fig. 7c) 90$^{o}$ counterclockwise and you get Fig. 7f. Apply the Corner Piece Series and the blue color of the orange/blue/yellow corner piece moves next to the blue color of the blue/orange edge piece. Turn the orange face 90$^{o}$ clockwise and two corner pieces have been moved into place.**





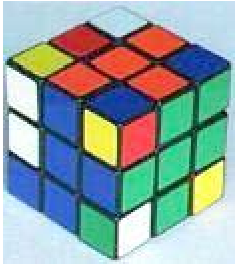
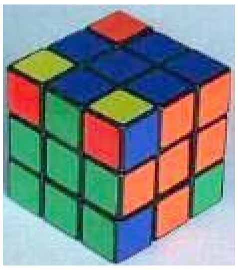


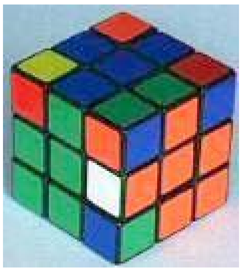
Fig. 7f Fig. 8a Fig. 8b

**Fig. 8a shows another example. The orange/blue/green corner piece at back/top/right will move along the top/right edge of the cube and "roll over" into proper position/orientation in the front/top/right corner of the cube. It replaces the yellow/blue/red corner piece. What needs to be done to cause the latter piece to move into position/orientation at the same time?**

**According to its colors the yellow/blue/red piece belongs in the corner diagonally across the blue face of the cube. Rotate the yellow (left) face of the cube 180$^{o}$ giving Fig. 8b. Apply the Corner Piece Series (first turn of the top face is to the left) and the yellow/blue/red piece moves into place with its red color next to the red edge piece color on the left side of the top face. A second 180$^{o}$ turn of the left face and two corner pieces have moved into place during the same sequence of moves.**

**Fig. 9a shows a potential problem. A Corner Piece Series (first turn of the top face is to the left) moves the red/yellow/blue piece across the blue (top) face of the cube to its correct position and orientation. But it would also move the yellow/blue/orange out of place. To avoid this, turn the orange face 90$^{o}$ clockwise (giving the cube in Fig. 9b) and apply a Corner Piece Series which begins with a turn of the top face to the right. Reverse the preliminary turn of the orange face. The red/yellow/blue piece winds up in the correct place but the yellow/blue/orange piece is not disturbed.**





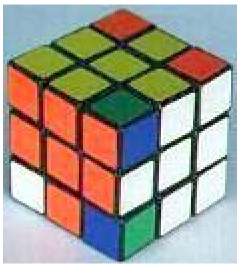
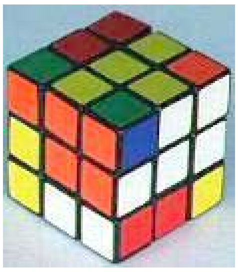


**Fig. 9a** **Fig. 9b** **Fig. 10a**

**We could move three corner pieces on the top face of the cube in Fig. 10a about the triangle: front/top/right; back/top/left; back/top/right. The yellow/orange/white piece at back/top/right moves along the top/right edge and rolls over into place at front/top/right. The blue color of orange/green/blue comes to the top in the back/top/left corner. This piece belongs in the front/bottom/left corner. The blue color of the orange/green/blue piece obviously must go in the blue (left) face of the cube. It must replace the red color of the piece which is now there.**

**Turn the bottom face 90$^o$ counterclockwise and then turn the left face 90$^o$ clockwise. This gives the cube in Fig. 10b. A Corner Piece Series which begins with a turn of the top face to the left causes the blue color of the orange/green/blue piece to replace the red color we see in the back/top/left corner of the cube. Then reverse the two preliminary turns (left face 90$^o$ counterclockwise, bottom face 90$^o$ clockwise). Again we have placed two corner pieces during the same sequence.**



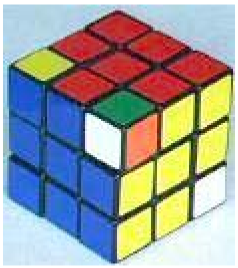
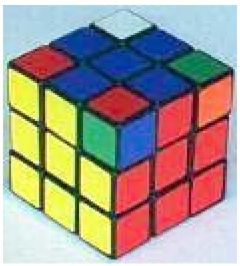


**Fig. 10b** **Fig. 11**

**The blue/yellow/red corner piece at front/top/left in Fig. 11 could move along the front/top edge and roll over into place at front/top/right. But we do not have a triangle of three out-of-place corner pieces on the top face. No problem. Rotate the white face 90$^o$ clockwise and the out-of-place piece at back/bottom/right comes to the top face making a temporary triangle. Apply the Corner Piece Series and then turn the white face 90$^o$ counterclockwise.**

**The yellow/red/blue piece at front/top/left in Fig. 12a will move along the front/top edge and into place at front/top/right. The green/red/blue piece has the most difficult situation you will find in trying to place two corner pieces at the same time. This is indicated by the checkerboard appearance it makes with the red/blue edge piece next to it. This corner piece belongs at back/top/right and its green face must be in the green face of the cube.**



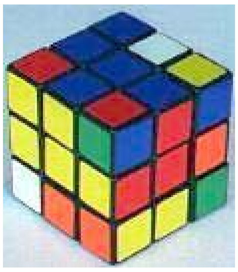


**Fig. 12a** **Fig. 12b**

**It will replace the blue/green/orange corner piece which now has its blue face in the green face of the cube. This means we must find (create) a Corner Piece Series in which the green face of the green/red/blue piece replaces the blue face of the blue/orange/green piece. Turn the green**

face 90$^o$ counterclockwise; turn the white (bottom) face 90$^o$ clockwise; and turn the green face 90$^o$ counterclockwise. We have the cube in Fig. 12b. As required, the blue face we are looking for is on the top face at back/top/left.

Apply a Corner Piece Series moving corner pieces about the triangle: front/top/left; front/top/right; back/top/left. Now reverse the preliminary turns (green face 90$^o$ clockwise; white face 90$^o$ counterclockwise; green face 90$^o$ clockwise). Again, two corner pieces have been moved into place at the same time.

The cube in Fig. 13 shows an unusual arrangement. The pieces making a checkerboard pattern on opposite sides of the red/yellow edge piece need to be exchanged as do the pieces across the diagonal on the white face. We could put one of the first pair in place using the preliminary three turn sequence described in the previous paragraph. Or we could put one of the latter pair in place using a two turn sequence. We will do the latter simply because it is shorter.

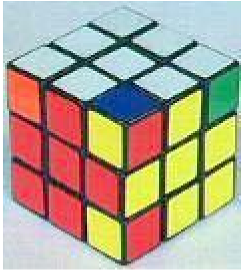


Fig. 13

Move the green/red/white corner piece so that it is across the cube diagonal (not a face diagonal) from its proper position (turn the orange face 90$^o$ counterclockwise). Next turn the blue (bottom) face 90$^o$ counterclockwise. Note that this turn is at right angles to the first turn. Now the green/red/white corner piece is diagonally across the red face from its proper position and can be moved into position and orientation with the proper Corner Piece Series.

If the second turn in the paragraph above had been 180$^o$ instead of 90$^o$ then the green/red/white piece could have been moved into place along a cube edge (the green/red edge) with the proper Corner Piece Series.

When five corner pieces have been placed Step Four will be completed. If you can place two corner pieces at a time obviously the total number of applications of the Corner Piece Series will be reduced. However, anytime you find a piece in place but with the wrong orientation you will have to remove it. You can do this at the same time that you move another corner piece into place but you can't place two pieces while removing a third. And anytime you find two corner pieces are each located in the other's proper place you will be able to place only one corner piece while moving the other out of the way.

## STEP FIVE: THE END GAME

With only three corner pieces remaining out of position all must be involved in the final Corner Piece Series. Further, each must move simultaneously into its correct position and with the right orientation.

It is possible that the three target corner pieces are on the same face and will move into place via a Corner Piece Series without any preliminary moves. But that would be rare indeed. We won't even think of it. Rather, the first thing for you to do is to look for a corner piece which can move along a cube edge into its correct position while rolling over into its correct orientation. Surprising as it may seem, in about half of all instances you will find that such a piece exists. About one-fourth of the time you will even find two such pieces.

**The corner piece at back/top/right in Fig. 14a is just such a piece. (Its color on the back side of the cube is blue.) The proper Corner Piece Series will cause it to move along the top/right edge to the front/top/right corner of the cube where it will be in the correct position/orientation. There are two different cube faces which may be used as the front face in the Corner Piece Series and two different triangles about which the corner pieces may move.**

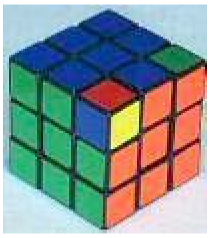


**Fig. 14a**

**In one such case, the blue/red/yellow piece at front/top/right will move to the back/top/left corner (i.e. the yellow/blue/red corner) with its yellow face on top of the cube. In that series, the green face is front and the first move is a rotation of the blue (top) face to the left.**

**In the other case, the front face is red (the left side in Fig. 14a) and the first turn is again a rotation of the blue (top) face to the left. In this case the blue/red/yellow piece will move to the corner now occupied by the blue/green/red corner piece. Again the yellow face of the blue/red/yellow corner piece will come to the top of the cube.**

**You should note that the first two corner pieces will be on the same face and will be adjacent to each other. The first will go into position/orientation replacing the second. The second will go to either of the two remaining corners on the top face with the color which is on the right side of the cube at the start of the Corner Piece Series coming to the top. We will refer to the third and fourth corners as the "target positions".**

**Naturally the second corner piece must replace the third corner piece with the third replacing the first. But where is the third corner piece? A little thought should convince you that, in this case, the third piece is somewhere in the yellow face of the cube. Fig. 14b (the entire cube in Fig. 14a has been rotated 90$^o$ clockwise on an axis from top to bottom) shows that the third corner piece is in the yellow/blue/red corner. It should be clear that if the yellow face of the second corner piece replaces the orange face of the third corner piece (the face which is in the yellow face of the cube) the second piece would be in the right position and have the right orientation.**

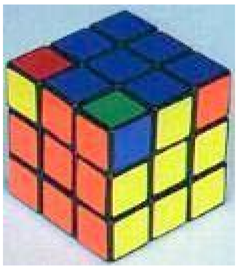


**Fig. 14b**

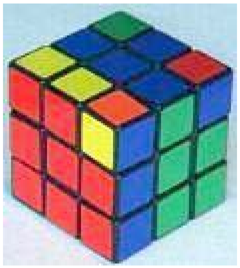


**Fig. 14c**

**How can we make that happen? We must bring that orange face to the top (blue) face of the cube in one of the target positions. This can be done with a 90$^o$ clockwise rotation of the red face as shown in Fig. 14c. Then, with red as the front face, a Corner Piece Series beginning with a turn of the blue face to the left will cause these three corner pieces to move clockwise about the front/top/right; back/top/left; back/top/right triangle.**

**The yellow face of the yellow/red/blue corner piece replaces the orange face of the orange/blue/yellow corner piece and when the red face is rotated 90$^o$ counterclockwise (the preliminary turn is reversed) each of the corner pieces will be in its correct position and orientation.**

**But, you might say,..."You made sure that each of the first two pieces went to the right place but you did nothing about the third". You don't have to worry about the third corner piece for, when all other pieces on the cube have been properly placed, the last piece must be in position and must be properly oriented as well. It is physically impossible for it to be otherwise. Thus you must ensure that the first two pieces are correctly placed and the third one will take care of itself.**

**As indicated above, when you reach the End Game, half the time at least one of the corner pieces will be in position such that a Corner Piece Series will cause it to move along a cube edge and into position/orientation. If you don't find such a corner piece you must create one. About 70% of the time you will do so with a single 90$^o$ rotation of a cube face. And about 30% of the time you will do so with a 180$^o$ rotation of a cube face.**

**The cube in Fig. 15a does not have a corner piece which can be moved along a cube edge and into place. But the red/white/yellow corner piece is diagonally across the yellow face from the corner where it belongs. It could be made part of a triangle which would send it across the diagonal to its proper place. A corner piece in such a position may be moved to an adjacent corner so as to create a corner piece which can be moved along a cube edge and into place. You can do this with a 90$^o$ rotation of either vertical face in which the corner piece is located. For example, the orange face could be rotated 90$^o$ clockwise. Fig. 15b shows the resulting cube with white as the front face.**

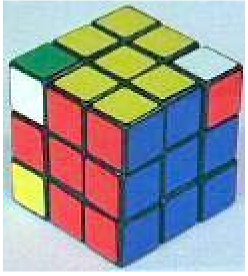


**Fig. 15a**

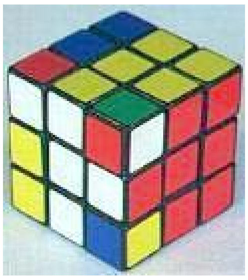


**Fig. 15b**

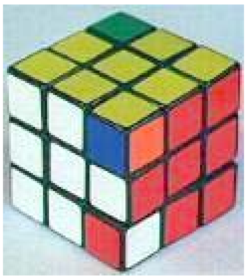


**Fig. 16**

**The cube in Fig. 16 does not have an appropriate corner piece adjacent to its correct position. Neither does it have a corner piece which could be moved across a face diagonal to its proper place. But it does have a corner piece which can be rotated 180$^o$ and be in a position from which it can move along a cube face and into proper position/orientation. That corner piece in Fig. 16 is the green/red/white piece in the back/top/left corner of the cube (all you can see is the green face).**

**You can turn either of the faces, which (of the three remaining out-of-position corner pieces) contain only the green/red/white corner piece, by 180$^o$ and that piece will be in position such that the proper Corner Piece Series will move it along a cube edge and into place. Turning the third face of the cube (yellow in this case) will not work.**

**With this understanding we can construct the rules for successfully completing the End Game:**

**1. Find a corner piece which may be moved along a cube edge and into place. If none can be found create one.**

**2. Note the face color (of the second corner piece) which will come to the top face of the cube in a target position during a Corner Piece Series.**

**3. Locate the third corner piece in the cube face of this same color and note the color of the third corner piece face which is in that cube face.**

**4. Bring the third corner piece to the top of the cube, by various face turns, such that that color is in the top face of the cube.**

5. Perform the Corner Piece Series so that the target color of the second corner piece replaces this color (named in 4 above).

6. Reverse the preliminary face turns indicated in 4 above.

This process is much easier to carry out than it is to describe. Several examples will be given below.

The white/yellow/orange corner piece at front/right/top of the cube in Fig. 17a may be moved along the white/yellow edge and into place. It will replace the red/green/blue piece. The latter will move to either of the target positions with its blue face coming to the top. We examine the blue face of the cube and find the yellow face of the white/red/yellow corner piece in that blue face. Hence we need to get the yellow face of the third corner piece into one of the two target positions. This is done by a 180$^{o}$ rotation of the green face giving the cube in Fig. 17b (the entire cube has been rotated 180$^{o}$ so you can see all three faces of the third corner piece).

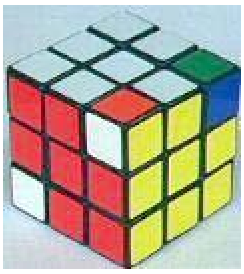


Fig. 17a

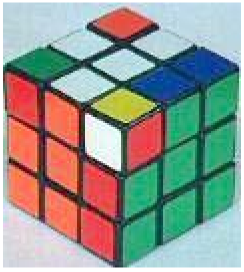


Fig. 17b

Now hold the cube with the green face in front and begin the Corner Piece Series with a turn of the top (white) face to the right. The third corner piece goes across the diagonal of the white face (with its white face coming to the top) and into its proper position and orientation.

You do not always have to examine the face of the cube which has the target color. When the white/yellow/red piece in Fig. 18a moves along the white/yellow edge of the cube the green face of the second corner piece comes to the top in either of the target positions. We need to move the face of the piece at the lower left of the cube which is now in the green face (we see only its white face) into one of the target positions.

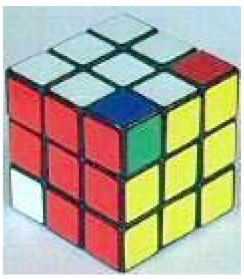


Fig. 18a

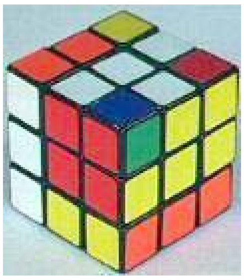


Fig. 18b

Rotate the bottom face by 90$^{o}$ counterclockwise and then the left face 90$^{o}$ clockwise. This gives Fig. 18b. The face we were looking for is yellow (we would have seen that if we had turned the cube to reveal the green face). The first turn of the top face is a rotation to the top left. When the series is completed reverse the original two turns and the cube is complete.

The corner piece at back/top/left in Fig. 19a will move along the top/left edge of the cube and into place. The green face of the white/orange/green corner piece will come to the top in a target position. The yellow face of the third corner piece is in the green face of the cube. We can bring that yellow face to a target position by the three turn sequence we learned in Step Four. It is: green counterclockwise; blue (bottom) clockwise; and green counterclockwise.

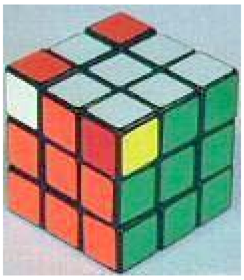


Fig. 19a

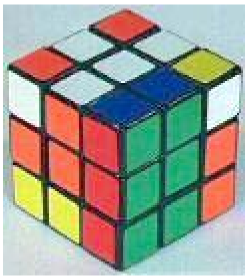


Fig. 19b

**There is the yellow color on the top face in the back/right corner of the cube (Fig. 19b). Apply the Corner Piece Series and reverse the three preliminary turns. The cube is restored.**

**The cube in Fig. 20a has two corner pieces which can move along a cube edge and into position. We could use either one of them as the first corner piece. Note that all three of the final corner pieces have an orange face and, in each case it is in the orange face of the cube.**

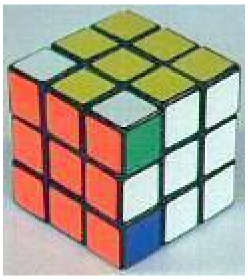


Fig. 20a

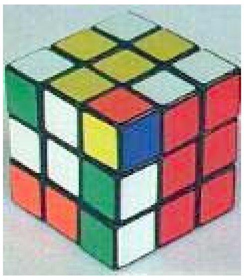


Fig. 20b

**We will use the yellow/white/orange piece as the first corner piece. Let us turn the entire cube 90$^o$ clockwise on an axis from top to bottom. Now turn the bottom (green) face 90$^o$ clockwise and then the red face also 90$^o$ clockwise. This gives the cube in Fig. 20b. (Again we chose this view so you could see all three faces of the third corner piece.) With red as front, turn the top face to the right and complete the Corner Piece Series. Reverse the two preliminary turns and the cube will be done.**

**In Fig. 21a none of the pieces is adjacent to any other. As a general rule this should be avoided but it is not much of a problem in this case because any one of the three corner pieces could be turned 90$^o$ and thereby create a corner piece which could be moved along a cube edge and into its proper place. For example, turn the orange face 90$^o$ counterclockwise (Fig. 21b shows the result except that the entire cube is turned clockwise in order to show all three faces of the green/white/red corner piece) or turn the red face 90$^o$ clockwise.**

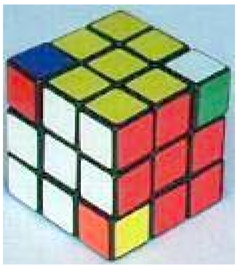


Fig. 21a

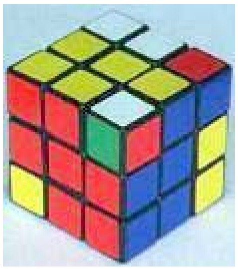


Fig. 21b

**But if the orientation of each were to be changed so as to put a corner piece face into the cube face of the same color then things would change drastically. For example, the yellow color of the red/blue/yellow piece is moved to the yellow face of the cube at front/top/left, the red face of red/white/green is moved to the red face of the cube and the white face of orange/white/yellow is moved to the white face of the cube. Although this arrangement is rare, it is the most difficult you will encounter. I will leave this for you to solve for yourself. But I'll give you a hint. You will have to move one of the corner pieces in two different directions.**

## Cube Scrambling Algorithms

1) F' U L F D' B2 U2 D2 R2 D2 L2 R2 D L' F2 L2 D B D R' B R2 L D L'
2) B' F U' R2 D U B2 U R D' L2 U' B U L' D2 B' D' U2 L U B L' U B
3) L R2 D2 R' D2 L2 R2 F' B L' B' U2 D B U2 B' D2 L2 R U2 R' B2 L R2 D
4) B2 F2 D' R' B' R F R L2 F' U B2 F R2 F U' B R' U D' L F2 D' U L
5) B' R2 B' F2 U' R' F R2 D2 U' B D L2 U R F2 B' D2 R U2 L D2 F' U2 B2

## Superflip Algorithm

20 Moves
U R2 F B R B2 R U2 L B2 R U' D' R2 F R' L B2 U2 F2